

Partition Functions for Maxwell Theory on the Five-torus
and for the Fivebrane on $S^1 \times T^5$

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Abstract

We compute the partition function of five-dimensional abelian gauge theory on a five-torus T^5 with a general flat metric using the Dirac method of quantizing with constraints. We compare this with the partition function of a single fivebrane compactified on S^1 times T^5 , which is obtained from the six-torus calculation of Dolan and Nappi [[arXiv:hep-th/9806016](https://arxiv.org/abs/hep-th/9806016)]. The radius R_1 of the circle S^1 is set to the dimensionful gauge coupling constant $g_{5YM}^2 = R_1$.

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1 Introduction

A quantum equivalence between the six-dimensional $N = (2, 0)$ theory of multiple fivebranes compactified on a circle S^1 and five-dimensional maximally supersymmetric Yang Mills has been conjectured by Douglas and Lambert *et al.* in [1, 2]. In this paper we will study an abelian version of the conjecture where the common five-manifold is a five-torus T^5 with a general flat metric.

The physical degrees of freedom of a single fivebrane are described by an $N = (2, 0)$ tensor supermultiplet which includes a chiral two-form field potential, so even a single fivebrane has no fully covariant action. In order to investigate its quantum theory we were thus led in [3] to compute the partition function instead, which we carried out on the six-torus T^6 . We will use this calculation to investigate the partition function of the self-dual three-form field strength restricted to $S^1 \times T^5$ and compare it with the partition function of the five-dimensional Maxwell theory on a five-torus quantized via Dirac constraints in radiation gauge.

Because both the theory and the manifold are so simple, we do not use localization techniques fruitful for non-abelian theories and their partition functions on spheres [4]-[9].

Our calculation differs from Gustavsson [10] who also considers the conjecture on a torus but uses holomorphic factorization and a supersymmetric five-dimensional abelian gauge theory whose partition function immediately has zero oscillator trace. See also [11].

The five-dimensional Maxwell partition function is defined from the Euclidean action as

$$Z^{5d, Maxwell} \equiv \text{tr} e^{-I^{5d, Maxwell}} = Z_{\text{zero modes}}^{5d} \cdot Z_{\text{osc}}^{5d},$$

$$I^{5d, Maxwell} = \int_0^{2\pi} d\theta^2 \dots d\theta^6 \frac{\sqrt{\tilde{G}_5}}{4g_{5YM}^2} \tilde{G}_5^{\tilde{m}\tilde{m}'} \tilde{G}_5^{\tilde{n}\tilde{n}'} F_{\tilde{m}\tilde{n}}(\theta^2, \dots, \theta^6) F_{\tilde{m}'\tilde{n}'}(\theta^2, \dots, \theta^6). \quad (1.1)$$

The partition function of the abelian chiral two-form on a space circle times the five-torus is

$$Z^{6d, chiral} = \text{tr} e^{-2\pi R_6 \mathcal{H} + i2\pi \gamma^i P_i} = Z_{\text{zero modes}}^{6d} \cdot Z_{\text{osc}}^{6d},$$

$$\mathcal{H} = \frac{1}{12} \int_0^{2\pi} d\theta^1 \dots d\theta^5 \sqrt{G_5} G_5^{ll'} G_5^{mm'} G_5^{nn'} H_{lmn}(\vec{\theta}, \theta^6) H_{l'm'n'}(\vec{\theta}, \theta^6),$$

$$P_i = -\frac{1}{24} \int_0^{2\pi} d\theta^1 \dots d\theta^5 \epsilon^{rsumn} H_{umn}(\vec{\theta}, \theta^6) H_{irs}(\vec{\theta}, \theta^6) \quad (1.2)$$

where θ^1 is the direction of the circle S^1 . The time direction θ^6 we will use for quantization is common to both theories, and the angles between the circle and the five-torus denoted by α, β^i in [3] have been set to zero.

We will compute an exact equivalence (up to an overall factor) between the zero mode contributions,

$$Z_{\text{zero modes}}^{6d} = Z_{\text{zero modes}}^{5d} \cdot \frac{\sqrt{g}}{(R_1 R_6)^2}. \quad (1.3)$$

Not surprisingly, we find the oscillator traces differ by the absence in Z_{osc}^{5d} of the Kaluza-Klein modes generated in Z_{osc}^{6d} from compactification on the circle S^1 . Also we find the Maxwell trace is independent of γ^i , the angles of the four space directions with the time axis.

The Kaluza-Klein modes have been associated with instantons in the five-dimensional non-abelian gauge theory in [1, 2, 12, 13], with additional comments given for the abelian limit. It would be interesting to find a systematic way to incorporate these modes in a generalized five-dimensional partition function along the lines of a character, in order to match the partition functions exactly, but we have not done that here. Rather our explicit expressions show a similarity between the oscillator traces of the two theories in the limit where the compactification radius R_1 is small compared to the five-torus T^5 .

Other approaches to $N = (2, 0)$ theories formulate fields for non-abelian chiral two forms [14]-[18] which would be useful if the non-abelian six-dimensional theory has a classical description and if the quantum theory can be described in terms of fields. On the other hand the partition functions on various manifolds [19]-[24] can demonstrate aspects of the six-dimensional finite quantum conformal theory presumed responsible for features of four-dimensional gauge theory [25].

In section 2, the contribution of the zero modes to the partition function for the chiral theory on a circle times a five-torus is computed as a sum over the ten integer eigenvalues, and its relation to that of the gauge theory is shown via a Poisson resummation. In section 3, the abelian gauge theory is quantized on a five-torus using Dirac constraints, and the Euclidean action is computed in terms of the oscillator modes. In section 4, we construct the oscillator trace contribution to the partition function for the gauge theory and compare it with that of the chiral two form. Section 5 contains discussion and conclusions. Appendix A presents details of the Dirac quantization and Appendix B describes a modified Hamiltonian, leading to the same commutation relations. Appendix C regularizes the vacuum energy.

2 Zero Modes

The $N = (2, 0)$ 6d world volume theory of the fivebrane contains five scalars, two four-spinors and a chiral two-form B_{MN} , which has a self-dual three-form field strength $H_{LMN} = \partial_L B_{MN} + \partial_M B_{NL} + \partial_N B_{LM}$ with $1 \leq L, M, N \leq 6$,

$$H_{LMN}(\vec{\theta}, \theta^6) = \frac{1}{6\sqrt{-G}} G_{LL'} G_{MM'} G_{NN'} \epsilon^{L'M'N'RST} H_{RST}(\vec{\theta}, \theta^6). \quad (2.1)$$

(2.1) gives $H_{LMN}(\vec{\theta}, \theta^6) = \frac{i}{6\sqrt{|G|}} G_{LL'} G_{MM'} G_{NN'} \epsilon^{L'M'N'RST} H_{RST}(\vec{\theta}, \theta^6)$ for a Euclidean signature metric. In the absence of a covariant Lagrangian, the partition function of the chiral field is defined via a trace over the Hamiltonian [3] as is familiar from string calculations. We display this expression in (1.2) where the metric has been restricted to describe the line

element for $S^1 \times T^5$,

$$ds^2 = R_1^2 (d\theta^1)^2 + R_6^2 (d\theta^6)^2 + \sum_{i,j=2\dots 5} g_{ij} (d\theta^i - \gamma^i d\theta^6)(d\theta^j - \gamma^j d\theta^6) \quad (2.2)$$

with $0 \leq \theta^I \leq 2\pi$, $1 \leq I \leq 6$. The parameters R_1 and R_6 are the radii for directions 1 and 6, g_{ij} is a 4d metric, and γ^j are the angles between between 6 and j . So from (2.2),

$$G_{ij} = g_{ij}; \quad G_{11} = R_1^2; \quad G_{i1} = 0; \quad G_{66} = R_6^2 + g_{ij} \gamma^i \gamma^j; \quad G_{i6} = -g_{ij} \gamma^j; \quad G_{16} = 0; \quad (2.3)$$

and the inverse metric is

$$G^{ij} = g^{ij} + \frac{\gamma^i \gamma^j}{R_6^2}; \quad G^{11} = \frac{1}{R_1^2}; \quad G^{1i} = 0; \quad G^{66} = \frac{1}{R_6^2}; \quad G^{i6} = \frac{\gamma^i}{R_6^2}; \quad G^{16} = 0. \quad (2.4)$$

We want to keep the time direction θ^6 common to both theories, so in the 5d action (1.1) the sum is on $2 \leq \tilde{m}, \tilde{n} \leq 6$; whereas the Hamiltonian in (1.2) is independent of time and sums on $1 \leq m, n \leq 5$. The common space index is labeled $2 \leq i, j \leq 5$. To this end, for the metric G_{MN} in (2.3) we introduce the 5-dimensional inverse (in directions 1,2,3,4,5)

$$G_5^{ij} = g^{ij}; \quad G_5^{i1} = 0; \quad G_5^{11} = \frac{1}{R_1^2}; \quad (2.5)$$

and the 5-dimensional inverse (in directions 2,3,4,5,6) for the five-torus T^5 ,

$$\tilde{G}_5^{ij} = g^{ij} + \frac{\gamma^i \gamma^j}{R_6^2}; \quad \tilde{G}_5^{i6} = \frac{\gamma^i}{R_6^2}; \quad \tilde{G}_5^{66} = \frac{1}{R_6^2}. \quad (2.6)$$

The determinants of the metrics are related simply by $\sqrt{G} = R_6 \sqrt{G_5} = R_1 \sqrt{\tilde{G}_5} = R_6 R_1 \sqrt{g}$, and $\epsilon_{23456} \equiv \tilde{G}_5 \epsilon^{23456} = \tilde{G}_5$, with corresponding epsilon tensors related by G, G_5, g .

To compute $Z_{\text{zero modes}}^{6d}$ we neglect the integrations in (1.2) and get

$$\begin{aligned} -2\pi R_6 \mathcal{H} &= -\frac{\pi}{6} R_6 R_1 \sqrt{g} g^{ii'} g^{jj'} g^{kk'} H_{ijk} H_{i'j'k'} - \frac{\pi}{4} \frac{R_6}{R_1} \sqrt{g} (g^{jj'} g^{kk'} - g^{jk'} g^{kj'}) H_{1jk} H_{1j'k'}, \\ i2\pi \gamma^i P_i &= -\frac{i\pi}{2} \gamma^i \epsilon^{jkj'k'} H_{1jk} H_{ij'k'} = \frac{i\pi}{3} \epsilon^{jj'kk'} H_{j'kk'} H_{1ij}, \end{aligned} \quad (2.7)$$

where the zero modes of the four fields H_{ijk} are labeled by the integers n_7, \dots, n_{10} . The six fields H_{1jk} have zero mode eigenvalues $H_{123} = n_1$, $H_{124} = n_2$, $H_{125} = n_3$, $H_{134} = n_4$, $H_{135} = n_5$, $H_{145} = n_6$, and the trace on the zero mode operators in (1.2) is

$$\begin{aligned}
Z_{\text{zero modes}}^{6d} &= \sum_{n_1, \dots, n_6} \exp \left\{ -\frac{\pi R_6}{4 R_1} \sqrt{g} (g^{jj'} g^{kk'} - g^{jk'} g^{kj'}) H_{1jk} H_{1j'k'} \right. \\
&\quad \cdot \sum_{n_7, \dots, n_{10}} \exp \left\{ -\frac{\pi}{6} R_6 R_1 \sqrt{g} g^{ii'} g^{jj'} g^{kk'} H_{ijk} H_{i'j'k'} - \frac{i\pi}{2} \gamma^i \epsilon^{jkj'k'} H_{1jk} H_{ij'k'} \right\}.
\end{aligned} \tag{2.8}$$

The same sum is obtained from the 5d Maxwell theory (1.1) where the gauge coupling is identified with the radius squared of the circle $g_{5YM} = R_1^2$, as follows. Letting $\int d\theta^6 = 2\pi$ and neglecting the $d\theta^i$ space integrations we find the action gives

$$\begin{aligned}
&- I^{5d, \text{Maxwell}} \\
&= -\frac{\pi R_6}{2 R_1} \sqrt{g} g^{jj'} g^{kk'} F_{jk} F_{j'k'} - \frac{\pi \sqrt{g}}{R_1 R_6} \gamma^j \gamma^{j'} g^{kk'} F_{jk} F_{j'k'} - \frac{2\pi \sqrt{g}}{R_1 R_6} g^{kk'} \gamma^i F_{ik} F_{6k'} - \frac{\pi \sqrt{g}}{R_1 R_6} g^{jj'} F_{6j} F_{6j'}.
\end{aligned} \tag{2.9}$$

Then the Maxwell zero mode trace can be written as

$$\begin{aligned}
Z_{\text{zero modes}}^{5d} &= \sum_{n_1 \dots n_6} \exp \left\{ -\frac{\pi R_6 \sqrt{g}}{2 R_1} g^{ii'} g^{jj'} F_{ij} F_{i'j'} \right\} \\
&\quad \cdot \sum_{n_7 \dots n_{10}} \exp \left\{ -\frac{\pi \sqrt{g}}{R_1 R_6} g^{jj'} [F_{6j} + \gamma^i F_{ij}] [F_{6j'} + \gamma^{i'} F_{i'j'}] \right\} \\
&= \sum_{n_1 \dots n_6} \exp \left\{ -\frac{\pi R_6 \sqrt{g}}{2 R_1} g^{ii'} g^{jj'} F_{ij} F_{i'j'} \right\} \sum_{n_7 \dots n_{10}} \exp \left\{ -\pi(n+x) \cdot A \cdot (n+x) \right\}
\end{aligned} \tag{2.10}$$

where the integer eigenvalues are $n_1 = F_{23}, n_2 = F_{24}, n_3 = F_{25}, n_4 = F_{34}, n_5 = F_{35}, n_6 = F_{45}, n_7 = F_{62}, n_8 = F_{63}, n_9 = F_{64}, n_{10} = F_{65}$, and

$$A^{jj'} = \frac{\sqrt{g}}{R_1 R_6} g^{jj'}, \quad x_j = \gamma^i F_{ij}. \tag{2.11}$$

Using a generalization of the Poisson summation formula

$$\sum_{n \in \mathbb{Z}^p} e^{-\pi(n+x) \cdot A \cdot (n+x)} = (\det A)^{-\frac{1}{2}} \sum_{n \in \mathbb{Z}^p} e^{-\pi n \cdot A^{-1} \cdot n} e^{2\pi i n \cdot x},$$

we obtain from (2.10),

$$\begin{aligned}
Z_{\text{zero modes}}^{5d} &= (\det A)^{-\frac{1}{2}} \sum_{n_1 \dots n_6} \exp\left\{-\frac{\pi}{2} \frac{R_6 \sqrt{g}}{R_1} g^{ii'} g^{jj'} F_{ij} F_{i'j'}\right\} \\
&\cdot \sum_{n_7 \dots n_{10}} \exp\left\{-\pi \frac{R_1 R_6}{\sqrt{g}} g_{jj'} F_6^j F_6^{j'}\right\} \exp\{2\pi i F_6^j \gamma^i F_{ij}\}, \quad (2.12)
\end{aligned}$$

where the F_6^j are defined to be the integers F_{6j} and

$$A_{jj'}^{-1} = \frac{R_1 R_6}{\sqrt{g}} g_{jj'}, \quad \det A^{-1} = (\det A)^{-1} = \frac{(R_1 R_6)^4}{g}.$$

To demonstrate the equivalence between the zero mode sum for 5d Maxwell theory in (2.12) and the 6d chiral theory in (2.8), we can identify the integers as

$$H_{ijk} = -\frac{1}{g} \epsilon_{ijkk'} F_6^{k'}, \quad H_{1jk} = F_{jk},$$

so that the exponents are identified by

$$\begin{aligned}
-\frac{\pi}{6} R_6 R_1 \sqrt{g} g^{ii'} g^{jj'} g^{kk'} H_{ijk} H_{i'j'k'} &= -\pi \frac{R_1 R_6}{\sqrt{g}} g_{jj'} F_6^j F_6^{j'}, \\
-\frac{\pi}{4} \frac{R_6}{R_1} \sqrt{g} (g^{jj'} g^{kk'} - g^{jk'} g^{j'k}) H_{1jk} H_{1j'k'} &= -\frac{\pi}{2} \frac{R_6 \sqrt{g}}{R_1} g^{ii'} g^{jj'} F_{ij} F_{i'j'}, \\
-\frac{i\pi}{2} \gamma^i \epsilon^{jkj'k'} H_{1jk} H_{i'j'k'} &= 2\pi i F_6^j \gamma^i F_{ij}. \quad (2.13)
\end{aligned}$$

Thus we have proved the relation (1.3)

$$Z_{\text{zero modes}}^{6d} = Z_{\text{zero modes}}^{5d} \cdot \frac{\sqrt{g}}{(R_1 R_6)^2} \quad (2.14)$$

and the explicit expression is given by (2.12) or (2.8).

3 Dirac Quantization of Maxwell Theory on a Five-torus

To evaluate the oscillator contribution to the partition function in (1.1), we will first quantize the abelian gauge theory on the five-torus with a general metric. The equation of motion is $\partial^{\tilde{m}} F_{\tilde{m}\tilde{n}} = 0$. For $F_{\tilde{m}\tilde{n}} = \partial_{\tilde{m}} A_{\tilde{n}} - \partial_{\tilde{n}} A_{\tilde{m}}$, a solution is given by a solution to

$$\partial^{\tilde{n}} \partial_{\tilde{n}} A_{\tilde{m}} = 0, \quad \partial^{\tilde{m}} A_{\tilde{m}} = 0. \quad (3.1)$$

These have a plane wave solution $A_{\tilde{m}}(\vec{\theta}, \theta^6) = f_{\tilde{m}}(k)e^{ik \cdot \theta} + (f_{\tilde{m}}(k)e^{ik \cdot \theta})^*$ when

$$\tilde{G}_5^{\tilde{m}\tilde{n}} k_{\tilde{m}} k_{\tilde{n}} = 0, \quad k^{\tilde{m}} f_{\tilde{m}} = 0. \quad (3.2)$$

Using the metric (2.6), and solving for k_6 from (3.2),

$$k_6 = -\gamma^i k_i - iR_6 \sqrt{g^{ij} k_i k_j}, \quad (3.3)$$

where $2 \leq i, j \leq 5$. Use the gauge invariance $f_{\tilde{m}} \rightarrow f'_{\tilde{m}} = f_{\tilde{m}} + k_{\tilde{m}} \lambda$ to fix $f'_6 = 0$, which is the gauge choice

$$A_6 = 0.$$

This reduces the number of components of $A_{\tilde{m}}$ from 5 to 4. To satisfy (3.2), we can use the $\partial^{\tilde{m}} F_{\tilde{m}6} = -\partial_6 \partial^i A_i = 0$ component of the equation of motion to eliminate f_5 in terms of the three f_2, f_3, f_4 ,

$$f_5 = -\frac{1}{p^5}(p^2 f_2 + p^3 f_3 + p^4 f_4),$$

leaving just three independent polarization vectors corresponding the physical degrees of freedom of the 5d one-form with Spin(3) content 3.

Since a Lorentzian signature metric is needed for quantum mechanics, we modify the metric on the five-torus (2.3), (2.6) to be

$$\begin{aligned} G_{Lij} &= g_{ij}; \quad G_{L66} = -R_6^2 + g_{ij} \gamma^i \gamma^j; \quad G_{Li6} = -g_{ij} \gamma^j; \\ G_L^{ij} &= g^{ij} - \frac{\gamma^i \gamma^j}{R_6^2}; \quad G_L^{66} = -\frac{1}{R_6^2}; \quad G_L^{i6} = -\frac{\gamma^i}{R_6^2}. \end{aligned} \quad (3.4)$$

Our procedure will be to compute the Dirac bracket commutation relations, and finally the Euclidean action in terms of oscillators with

$$k_6 = -\gamma^i k_i - R_6 \sqrt{g^{ij} k_i k_j}, \quad (3.5)$$

and then after the quantization rotate back to (3.3).

So, dropping prefactors for the moment, we write the Euclidean Lagrangian:

$$L_E = -\frac{1}{4} \tilde{G}_5^{\tilde{m}\tilde{m}'} \tilde{G}_5^{\tilde{n}\tilde{n}'} F_{\tilde{m}\tilde{n}} F_{\tilde{m}'\tilde{n}'} = -\frac{1}{4} (F^{ij} F_{ij} + 2F^{6i} F_{6i}) \quad (3.6)$$

$$= -\frac{1}{4} g^{ii'} g^{jj'} F_{ij} F_{i'j'} - \frac{1}{2R_6^2} g^{ii'} \gamma^j \gamma^{j'} F_{ij} F_{i'j'} - \frac{1}{R_6^2} g^{kk'} \gamma^i F_{ik} F_{6k'} - \frac{1}{2R_6^2} g^{jj'} F_{6j} F_{6j'}, \quad (3.7)$$

the Lorentzian Lagrangian:

$$L = -\frac{1}{4} g^{ii'} g^{jj'} F_{ij} F_{i'j'} + \frac{1}{2R_6^2} g^{ii'} \gamma^j \gamma^{j'} F_{ij} F_{i'j'} + \frac{1}{R_6^2} g^{kk'} \gamma^i F_{ik} F_{6k'} + \frac{1}{2R_6^2} g^{jj'} F_{6j} F_{6j'}, \quad (3.8)$$

and the Lorentzian Hamiltonian:

$$\begin{aligned} H &= \Pi^i \partial^6 A_i - \tilde{G}_L^{66} L = \Pi^i \tilde{G}_L^{66} F_{6i} + \Pi^i \tilde{G}_L^{6j} \partial_j A_i + \Pi^i \tilde{G}_L^{66} \partial_i A_6 - \tilde{G}_L^{66} L \\ &= \tilde{G}_L^{66} (-L_E) + \tilde{G}_L^{66} (\gamma^j \Pi^i \partial_i A_j + \Pi^i \partial_i A_6) \end{aligned} \quad (3.9)$$

where the conjugate momentum is

$$\Pi^i = \frac{\delta L}{\delta \partial_6 A_i} = -F^{6i} = -\tilde{G}_L^{66} g^{ii'} (F_{6i'} + \gamma^j F_{ji'}). \quad (3.10)$$

We will quantize the Maxwell field on the five-torus with a general flat metric (3.4) in radiation gauge, using Dirac constraints [26, 27]. The theory has a primary constraint $\Pi^6(\vec{\theta}, \theta^6) = 0$, which we call

$$\phi^1(\theta) = \Pi^6(\vec{\theta}, \theta^6) \approx 0. \quad (3.11)$$

We can express the Hamiltonian (3.9) in terms of the conjugate momentum as

$$H = \frac{1}{2} \tilde{G}_L^{66} \Pi^i (F_{6i} + \gamma^j F_{ji}) + \frac{1}{4} \tilde{G}_L^{66} g^{ii'} g^{jj'} F_{ij} F_{i'j'} + \tilde{G}_L^{66} \Pi^i \partial_i A_6 + \tilde{G}_L^{6j} \Pi^i \partial_i A_j, \quad (3.12)$$

where we used

$$L = -\frac{1}{4} (2F^{6i} F_{6i} + F^{ij} F_{ij}), \quad F^{ij} = \gamma^i F^{6j} - \gamma^j F^{6i} + g^{ii'} g^{jj'} F_{i'j'}. \quad (3.13)$$

From (3.10), H becomes

$$H = -\frac{g_{ii'}}{2} \Pi^i \Pi^{i'} + \frac{\tilde{G}_L^{66}}{4} g^{ii'} g^{jj'} F_{ij} F_{i'j'} + \tilde{G}_L^{66} \Pi^i \partial_i A_6 + \tilde{G}_L^{6j} \Pi^i \partial_i A_j. \quad (3.14)$$

We integrate the density H to find

$$H_{can} = \int d^4\theta \left(-\frac{g_{ii'}}{2} \Pi^i \Pi^{i'} + \frac{\tilde{G}_L^{66}}{4} g^{ii'} g^{jj'} F_{ij} F_{i'j'} - \tilde{G}_L^{66} \partial_i \Pi^i A_6 - \tilde{G}_L^{6j} \partial_i \Pi^i A_j \right), \quad (3.15)$$

where the last terms have been integrated by parts. The primary Hamiltonian is defined by

$$H_p = \int d^4\theta \left(-\frac{g_{ii'}}{2} \Pi^i \Pi^{i'} + \frac{\tilde{G}_L^{66}}{4} g^{ii'} g^{jj'} F_{ij} F_{i'j'} - \tilde{G}_L^{66} \partial_i \Pi^i A_6 - \tilde{G}_L^{6j} \partial_i \Pi^i A_j + \lambda_1 \Pi^6 \right), \quad (3.16)$$

with λ_1 as a Lagrange multiplier. In Appendix A, we use the Dirac method of quantizing with constraints for the radiation gauge conditions $A_6 \approx 0$, $\partial^i A_i \approx 0$, and find the equal time commutation relations (A.13), (A.14):

$$\begin{aligned}
[\Pi^j(\vec{\theta}, \theta^6), A_i(\vec{\theta}', \theta^6)] &= -i \left(\delta_i^j - \tilde{G}_L^{jj'} (\partial_i \frac{1}{\tilde{G}_L^{kk'} \partial_k \partial_{k'}} \partial_{j'}) \right) \delta^4(\theta - \theta'), \\
[A_i(\vec{\theta}, \theta^6), A_j(\vec{\theta}', \theta^6)] &= 0, \quad [\Pi^i(\vec{\theta}, \theta^6), \Pi^j(\vec{\theta}', \theta^6)] = 0.
\end{aligned} \tag{3.17}$$

Appendix B modifies the Hamiltonian (3.16) to give the correct equations of motion, and shows that this modification does not affect the constraints and leads back to (3.17). Using Appendix A we can check explicitly that Dirac brackets with a constraint vanish, for example

$$\begin{aligned}
\{\Pi^j(\theta), \partial^i A_i(\theta')\}_D &= \{\Pi^j(\theta), \tilde{G}_L^{ik} \partial_k A_i(\theta') - g_{ik} \gamma^k \Pi^i(\theta') + \tilde{G}_L^{i6} \partial_i A_6(\theta')\}_D \\
&= \tilde{G}_L^{jk} \frac{\partial}{\partial \theta^k} \delta^4(\theta - \theta') - \tilde{G}_L^{jl} \frac{\partial}{\partial \theta^l} \delta^4(\theta - \theta') = 0 = [\Pi^j(\theta), \partial^i A_i(\theta')],
\end{aligned} \tag{3.18}$$

and

$$[\partial_j \Pi^j(\theta), A_i(\theta')] = \partial_j \left(\delta_i^j - \tilde{G}_L^{jj'} (\partial_i \frac{1}{\tilde{G}_L^{kk'} \partial_k \partial_{k'}} \partial_{j'}) \right) \delta^4(\theta - \theta') = 0. \tag{3.19}$$

In $A_6 = 0$ gauge, the free quantum vector field on the torus is expanded as

$$A_i(\vec{\theta}, \theta^6) = \text{zero modes} + \sum_{\vec{k} \neq 0, \vec{k} \in \mathcal{Z}_4} (f_i^\kappa a_{\vec{k}}^\kappa e^{ik \cdot \theta} + f_i^{\kappa*} a_{\vec{k}}^{\kappa\dagger} e^{-ik \cdot \theta}),$$

where $1 \leq \kappa \leq 3$, $2 \leq i \leq 5$ and k_6 defined in (3.5). The sum is on the dual lattice $\vec{k} = k_i \in \mathcal{Z}_4 \neq \vec{0}$. Having computed the zero mode contribution in (2.12), here we consider

$$A_i(\vec{\theta}, \theta^6) = \sum_{\vec{k} \neq 0} (a_{\vec{k}i} e^{ik \cdot \theta} + a_{\vec{k}i}^\dagger e^{-ik \cdot \theta}), \tag{3.20}$$

with polarizations in $a_{\vec{k}i} = f_i^\kappa a_{\vec{k}}^\kappa$. From (3.17) the commutator in terms of the oscillators is

$$\int \frac{d^4 \theta d^4 \theta'}{(2\pi)^8} e^{-ik_i \theta^i} e^{-ik'_{i'} \theta'^{i'}} [A_i(\vec{\theta}, 0), A_j(\vec{\theta}', 0)] = [(a_{\vec{k}i} + a_{-\vec{k}i}^\dagger), (a_{\vec{k}'j} + a_{-\vec{k}'j}^\dagger)] = 0. \tag{3.21}$$

The conjugate momentum $\Pi^j(\vec{\theta}, \theta^6)$ in (3.10) is expressed in terms of $a_{\vec{k}i}, a_{-\vec{k}i}^\dagger$ by

$$\Pi^j(\vec{\theta}, \theta^6) = -\frac{i}{R_6^2} \sum_{\vec{k}} (g^{jj'} R_6 \sqrt{g^{ij} k_i k_j} + g^{ji'} \gamma^{j'} k_{i'}) (a_{\vec{k}j'} e^{ik \cdot \theta} - a_{\vec{k}j'}^\dagger e^{-ik \cdot \theta}). \tag{3.22}$$

Then taking the Fourier transform of $\Pi^j(\vec{\theta}, \theta^6)$ at $\theta^6 = 0$, we have

$$\int \frac{d^4 \theta}{(2\pi)^4} e^{-ik_i \theta^i} \Pi^j(\vec{\theta}, 0) = -\frac{i}{R_6^2} g^{jj'} R_6 \sqrt{g^{ij} k_i k_j} (a_{\vec{k}j'} - a_{-\vec{k}j'}^\dagger) - \frac{i}{R_6^2} g^{ji'} \gamma^{j'} k_{i'} (a_{\vec{k}j'} + a_{-\vec{k}j'}^\dagger). \tag{3.23}$$

From (3.23) and the commutators (3.17) and (3.21), we find

$$\begin{aligned} & \int \frac{d^4\theta d^4\theta'}{(2\pi)^8} e^{-ik_i\theta^i} e^{-ik'_{i'}\theta'^{i'}} [\Pi^j(\vec{\theta}, 0), A_i(\vec{\theta}', 0)] \\ &= -i(\delta_i^j - \frac{\tilde{G}_L^{jj'} k_i k_{j'}}{\tilde{G}_L^{kk'} k_k k_{k'}}) \delta_{\vec{k}, -\vec{k}'} \frac{1}{(2\pi)^4} = -\frac{i}{R_6^2} g^{jj'} R_6 \sqrt{g^{ij} k_i k_j} [(a_{\vec{k} j'} - a_{-\vec{k} j'}^\dagger), (a_{\vec{k}' i} + a_{-\vec{k}' i}^\dagger)]. \end{aligned} \quad (3.24)$$

To reach the oscillator commutator (3.30), we define

$$A_{\vec{k} i} \equiv a_{\vec{k} i} + a_{-\vec{k} i}^\dagger = A_{-\vec{k} i}^\dagger, \quad E_{\vec{k} i} \equiv a_{\vec{k} i} - a_{-\vec{k} i}^\dagger = -E_{-\vec{k} i}^\dagger. \quad (3.25)$$

$$a_{\vec{k} i} = \frac{1}{2}(A_{\vec{k} i} + E_{\vec{k} i}), \quad a_{\vec{k} i}^\dagger = \frac{1}{2}(A_{\vec{k} i}^\dagger + E_{\vec{k} i}^\dagger) = \frac{1}{2}(A_{-\vec{k} i} - E_{-\vec{k} i}). \quad (3.26)$$

Now inverting (3.24) we have

$$[E_{\vec{k} j}, A_{\vec{k}' i}] = \frac{R_6^2}{R_6 \sqrt{g^{ij} k_i k_j}} \frac{1}{(2\pi)^4} (g_{jj'} - \frac{g_{jj'} \tilde{G}_L^{i'j'} k_{i'} k_i}{\tilde{G}_L^{kk'} k_k k_{k'}}) \delta_{\vec{k}, -\vec{k}'}, \quad (3.27)$$

and from (3.23) and the relations (3.17) and (3.21),

$$\begin{aligned} [A_{\vec{k} i}, A_{\vec{k}' j}] &= 0, \quad [E_{\vec{k} i}, E_{\vec{k}' j}] = \frac{1}{g^{ij} k_i k_j} \frac{R_6 \sqrt{g^{ij} k_i k_j}}{R_6^2} (-\gamma^{j'} k'_{j'} [E_{\vec{k} i}, A_{\vec{k}' j'}] + \gamma^{j'} k_i [E_{\vec{k}' j}, A_{\vec{k} j'}]) \\ &= \frac{1}{(2\pi)^4} \delta_{\vec{k}, -\vec{k}'} \frac{1}{\tilde{G}_L^{kk'} k_k k_{k'}} (g_{ii'} \gamma^{i'} k_j + g_{jj'} \gamma^{j'} k_i - \frac{2k_i k_j \gamma^l k_l}{g^{i'j'} k_{i'} k_{j'}}). \end{aligned} \quad (3.28)$$

Using (3.26),

$$[a_{\vec{k} i}, a_{\vec{k}' j}^\dagger] = \frac{1}{4} ([A_{\vec{k} i}, A_{-\vec{k}' j}] - [E_{\vec{k} i}, E_{-\vec{k}' j}] - [A_{\vec{k} i}, E_{-\vec{k}' j}] + [E_{\vec{k} i}, A_{-\vec{k}' j}]), \quad (3.29)$$

together with (3.27), (3.28) we find the oscillator commutation relation

$$\begin{aligned} [a_{\vec{k} i}, a_{\vec{k}' j}^\dagger] &= \frac{R_6^2}{4R_6 \sqrt{g^{ij} k_i k_j}} \frac{1}{(2\pi)^4} (2g_{ij} - \frac{g_{ij'} \tilde{G}_L^{i'j'} k_{i'} k_j}{\tilde{G}_L^{kk'} k_k k_{k'}} - \frac{g_{jj'} \tilde{G}_L^{i'j'} k_{i'} k_i}{\tilde{G}_L^{kk'} k_k k_{k'}}) \delta_{\vec{k}, \vec{k}'} \\ &\quad - \frac{1}{4(2\pi)^4} \frac{1}{\tilde{G}_L^{kk'} k_k k_{k'}} (g_{ii'} \gamma^{i'} k_j + g_{jj'} \gamma^{j'} k_i - \frac{2k_i k_j \gamma^l k_l}{g^{i'j'} k_{i'} k_{j'}}) \delta_{\vec{k}, \vec{k}'}. \end{aligned} \quad (3.30)$$

Inspecting this commutator and noting that multiplication by k^i does not annihilate it, we

remark in an aside that $k^i a_{\vec{k}i} = 0$ does not hold as a ‘strong’ equation: for example,

$$g^{ik} k_k [a_{\vec{k}i}, A_{\vec{k}'j}] = \frac{g^{ik} k_k}{2} [E_{\vec{k}i}, A_{\vec{k}'j}] = 0, \quad (3.31)$$

using $g^{jk} k_k (g_{ji} - \frac{g_{jj'} \tilde{G}_L^{i'j'} k_{i'} k_i}{\tilde{G}_L^{kk'} k_k k_{k'}}) = 0$. But if we had $k^i a_{\vec{k}i} = 0$, then

$$\begin{aligned} g^{ik} k_k [a_{\vec{k}i}, A_{\vec{k}'j}] &= \gamma^i \frac{R_6 \sqrt{g^{ll'} k_l k_{l'}}}{R_6^2} [a_{\vec{k}i}, A_{\vec{k}'j}] = \gamma^i \frac{R_6 \sqrt{g^{ll'} k_l k_{l'}}}{2R_6^2} [E_{\vec{k}i}, A_{\vec{k}'j}] \\ &= \frac{\gamma^i}{2(2\pi)^4} (g_{ji} - \frac{g_{jj'} \tilde{G}_L^{i'j'} k_{i'} k_i}{\tilde{G}_L^{kk'} k_k k_{k'}}) = \frac{\gamma^i}{2(2\pi)^4} \frac{g^{ll'} k_l k_{l'}}{\tilde{G}_L^{kk'} k_k k_{k'}} \gamma^i (g_{ij} - \frac{k_i k_j}{g^{ll'} k_l k_{l'}}) \neq 0, \end{aligned} \quad (3.32)$$

from (3.27), contradicting (3.31). Nonetheless the Dirac brackets are still compatible with the constraints (such as the ‘weak’ equation $\partial^i A_i(\theta) \approx 0$), in the sense that anywhere the constraints appear they can be set to zero because their Dirac bracket with any variable vanishes. This discussion explains why it is okay that the commutator in (3.30) is not compatible with the strong transverse condition $k^i a_{\vec{k}i} = 0$, $k^i a_{\vec{k}i}^\dagger = 0$. Notice that in the diagonal metric theory the strong transverse condition *is* consistent with the Dirac commutator; but the non-diagonal metric provides an extra term in (3.23) proportional to $A_{\vec{k}i}$, leading to this effect.

We use the commutator (3.30) to proceed with the evaluation of the Euclidean action (this is reminiscent of computing the Hamiltonian in the standard case). From (3.7), we express the Euclidean action

$$\begin{aligned} I_{\text{Euclidean}} &\equiv \frac{1}{(2\pi)^4} \int_0^{2\pi} d\theta^2 d\theta^3 d\theta^4 d\theta^5 \tilde{G}_5^{\tilde{m}\tilde{m}'} G_5^{\tilde{n}\tilde{n}'} F_{\tilde{m}\tilde{n}} F_{\tilde{m}'\tilde{n}'} \\ &= \int \frac{d^4\theta}{(2\pi)^4} \left(\frac{2}{R_6^2} g^{ii'} F_{6i} F_{6i'} + \frac{2}{R_6^2} g^{ii'} \gamma^j (F_{ji} F_{6i'} + F_{6i'} F_{ji}) + \tilde{G}_5^{ii'} \tilde{G}_5^{jj'} F_{ij} F_{i'j'} \right) \end{aligned} \quad (3.33)$$

in terms of the normal mode expansion (3.20)

$$A_i(\vec{\theta}, \theta^6) = \sum_{\vec{k} \neq 0} (a_{\vec{k}i} e^{ik \cdot \theta} + a_{\vec{k}i}^\dagger e^{-ik \cdot \theta}), \quad F_{6i}(\vec{\theta}, \theta^6) = \sum_{\vec{k} \neq 0} ik_6 (a_{\vec{k}i} e^{ik \cdot \theta} - a_{\vec{k}i}^\dagger e^{-ik \cdot \theta}), \quad (3.34)$$

where now k_6 is given in (3.5), and also define

$$\tilde{k}_6 = \gamma^i k_i - R_6 \sqrt{g^{ij} k_i k_j}. \quad (3.35)$$

For the first and last terms in (3.33) we find

$$\begin{aligned}
& \int \frac{d^4\theta}{(2\pi)^4} \frac{2}{R_6^2} g^{ii'} F_{6i} F_{6i'} \\
&= \frac{2}{R_6^2} g^{ii'} \sum_{\vec{k} \neq 0} \left(-k_6 \tilde{k}_6 \left(a_{\vec{k}i}^\dagger a_{-\vec{k}i'} e^{-2iR_6 \sqrt{g^{ij} k_i k_j} \theta^6} - a_{\vec{k}i}^\dagger a_{-\vec{k}i'}^\dagger e^{2iR_6 \sqrt{g^{ij} k_i k_j} \theta^6} \right. \right. \\
&\quad \left. \left. + k_6^2 (a_{\vec{k}i}^\dagger a_{\vec{k}i'}^\dagger + a_{\vec{k}i}^\dagger a_{\vec{k}i'}) \right) \right),
\end{aligned}$$

and

$$\begin{aligned}
& \int \frac{d^4\theta}{(2\pi)^4} \tilde{G}_5^{ii'} \tilde{G}_5^{jj'} F_{ij} F_{i'j'} \\
&= 2(\tilde{G}_5^{ii'} \tilde{G}_5^{jj'} - \tilde{G}_5^{ij'} \tilde{G}_5^{ji'}) \\
&\quad \times \sum_{\vec{k} \neq 0} \left(k_j k_{j'} (a_{\vec{k}i}^\dagger a_{-\vec{k}i'} e^{-2iR_6 \sqrt{g^{ij} k_i k_j} \theta^6} + a_{\vec{k}i}^\dagger a_{-\vec{k}i'}^\dagger e^{2iR_6 \sqrt{g^{ij} k_i k_j} \theta^6}) + k_j k_{j'} (a_{\vec{k}i}^\dagger a_{\vec{k}i'}^\dagger + a_{\vec{k}i}^\dagger a_{\vec{k}i'}) \right).
\end{aligned} \tag{3.36}$$

using the delta function

$$\int \frac{d^4\theta}{(2\pi)^4} e^{i(k_i - k_i') \theta^i} = \delta_{\vec{k}, \vec{k}'}. \tag{3.37}$$

Similarly, the second term in the action (3.33) becomes

$$\begin{aligned}
& \int \frac{d^4\theta}{(2\pi)^4} \frac{2}{R_6^2} g^{ii'} \gamma^j (F_{ji} F_{6i'} + F_{6i} F_{ji'}) \\
&= \frac{1}{R_6^2} \sum_{\vec{k} \neq 0} \left((-4g^{ii'} \gamma^j \gamma^{j'} k_j k_{j'} + 2g^{i'j'} \gamma^i k_{j'} \tilde{k}_6 - 2g^{ij'} \gamma^{i'} k_j k_6) \right. \\
&\quad \times (a_{-\vec{k}i'} a_{\vec{k}i}^\dagger e^{-2iR_6 \sqrt{g^{ij} k_i k_j} \theta^6} + a_{-\vec{k}i'}^\dagger a_{\vec{k}i}^\dagger e^{+2iR_6 \sqrt{g^{ij} k_i k_j} \theta^6}) \\
&\quad \left. + (4g^{ii'} \gamma^j k_j k_6 - 2g^{i'j'} \gamma^i k_{j'} k_6 - 2g^{ij'} \gamma^{i'} k_j k_6) (a_{\vec{k}i}^\dagger a_{\vec{k}i'}^\dagger + a_{\vec{k}i}^\dagger a_{\vec{k}i'}) \right). \tag{3.38}
\end{aligned}$$

So computing the action (3.33) by adding the terms (3.36) and (3.38), we first show the θ^6 dependence cancels as follows. The coefficient of $a_{\vec{k}i}^\dagger a_{-\vec{k}i'} e^{-2iR_6 \sqrt{g^{ij} k_i k_j} \theta^6}$ in this sum is:

$$\begin{aligned}
& \frac{2}{R_6^2} g^{ii'} (-k_6 \tilde{k}_6) - \frac{4}{R_6^2} g^{ii'} \gamma^j \gamma^{j'} k_j k_{j'} + 2g^{i'j'} \gamma^i k_{j'} \tilde{k}_6 - 2g^{ij'} \gamma^{i'} k_j k_6 + 2(\tilde{G}_5^{ii'} \tilde{G}_5^{jj'} - \tilde{G}_5^{ij'} \tilde{G}_5^{ji'}) k_j k_{j'} \\
&= 2k^i \tilde{k}^{i'},
\end{aligned} \tag{3.39}$$

where for convenience $k^i, \tilde{k}^{i'}$ are defined via the Lorentzian signature metric (3.4),

$$k^i = g^{ij} k_j + \gamma^i \frac{R_6 \sqrt{g^{ij} k_i k_j}}{R_6^2}, \quad \tilde{k}^i = -g^{ij} k_j + \gamma^i \frac{R_6 \sqrt{g^{ij} k_i k_j}}{R_6^2}. \tag{3.40}$$

The term proportional to $a_{\vec{k}i}^\dagger a_{-\vec{k}i'}^\dagger e^{2iR_6\sqrt{g^{ij}k_ik_j}\theta^6}$ has the same coefficient. Since the strong transverse condition is not available $k^i a_{\vec{k}i}$, $k^i a_{\vec{k}i}^\dagger \neq 0$ (3.32), we demonstrate the time-independence of the Euclidean action as follows: consider the θ^6 dependence of

$$\begin{aligned} \int \frac{d^4\theta}{(2\pi)^4} 2\partial^i A_i(\theta)\partial^j A_j(\theta) &= 2 \sum_{\vec{k} \neq 0} \left(-k^i \tilde{k}^{i'} (a_{\vec{k}i} a_{-\vec{k}i'} e^{-2iR_6\sqrt{g^{ij}k_ik_j}\theta^6} + a_{\vec{k}i}^\dagger a_{-\vec{k}i'}^\dagger e^{2iR_6\sqrt{g^{ij}k_ik_j}\theta^6}) \right. \\ &\quad \left. + k^i k^{i'} (a_{\vec{k}i} a_{\vec{k}i'}^\dagger + a_{\vec{k}i'}^\dagger a_{\vec{k}i}) \right). \end{aligned} \quad (3.41)$$

This is the same time dependence as remains in (3.39). So we can replace (3.39) as

$$\begin{aligned} &2 \sum_{\vec{k} \neq 0} \left(k^i \tilde{k}^{i'} (a_{\vec{k}i} a_{-\vec{k}i'} e^{-2iR_6\sqrt{g^{ij}k_ik_j}\theta^6} + a_{\vec{k}i}^\dagger a_{-\vec{k}i'}^\dagger e^{2iR_6\sqrt{g^{ij}k_ik_j}\theta^6}) \right) \\ &= - \int \frac{d^4\theta}{(2\pi)^4} 2\partial^i A_i(\theta)\partial^j A_j(\theta) + 2 \sum_{\vec{k} \neq 0} k^i k^{i'} (a_{\vec{k}i} a_{\vec{k}i'}^\dagger + a_{\vec{k}i'}^\dagger a_{\vec{k}i}). \end{aligned} \quad (3.42)$$

Since $\partial^i A_i(\theta)$ is a gauge constraint and commutes with everything, we can set the integral term to zero, and then add to the time-independent portion of (3.33) this factor:

$$\begin{aligned} &2 \sum_{\vec{k}} k^i k^{i'} (a_{\vec{k}i} a_{\vec{k}i'}^\dagger + a_{\vec{k}i'}^\dagger a_{\vec{k}i}) \\ &= \sum_{\vec{k}} \left(2g^{ij}g^{i'j'} k_j k_{j'} + 2\gamma^i \gamma^{i'} \frac{g^{ll'} k_l k_{l'}}{R_6^2} + 2g^{ij}g^{i'j'} k_j \gamma^{i'} \frac{R_6 \sqrt{g^{ll'} k_l k_{l'}}}{R_6^2} + 2g^{i'j'} k_{j'} \gamma^i \frac{R_6 \sqrt{g^{ll'} k_l k_{l'}}}{R_6^2} \right) \\ &\quad \times (a_{\vec{k}i} a_{\vec{k}i'}^\dagger + a_{\vec{k}i'}^\dagger a_{\vec{k}i}). \end{aligned} \quad (3.43)$$

See (3.45). From (3.36) and (3.38) the time-independent coefficient of $(a_{\vec{k}i} a_{\vec{k}i'}^\dagger + a_{\vec{k}i'}^\dagger a_{\vec{k}i})$ is

$$\begin{aligned} &4g^{ii'}g^{jj'} k_j k_{j'} + \frac{2}{R_6^2} g^{i'j'} \gamma^i k_{j'} R_6 \sqrt{g^{ij} k_i k_j} + \frac{2}{R_6^2} g^{ijj'} \gamma^{i'} k_{j'} R_6 \sqrt{g^{ij} k_i k_j} \\ &\quad + \frac{2}{R_6^2} g^{jj'} \gamma^i \gamma^{i'} k_j k_{j'} - 2g^{ij}g^{i'j'} k_j k_{j'}. \end{aligned} \quad (3.44)$$

Then adding (3.43) to (3.44), and restoring the prefactors of the action, we have from (3.33) and (2.9) that the exponent of the oscillator trace for the partition function (1.1) is

$$I^{5d,Maxwell} = \frac{(2\pi)^5 R_6 \sqrt{g}}{R_1} \sum_{\vec{k} \neq 0} \left(k^i k^{i'} + (g^{ii'}g^{jj'} - g^{ij'}g^{ji'}) k_j k_{j'} \right) (a_{\vec{k}i} a_{\vec{k}i'}^\dagger + a_{\vec{k}i'}^\dagger a_{\vec{k}i}). \quad (3.45)$$

Restoring the prefactors to Π^i from (B.5), and to the commutator in (3.24), we have that (3.30) is rescaled to the dimensionless commutator

$$\begin{aligned}
[a_{\vec{k}i}, a_{\vec{k}'j}^\dagger] &= \left(\frac{R_1}{R_6\sqrt{g}}\right) \frac{R_6^2}{4R_6\sqrt{g^{ij}k_ik_j}} \frac{1}{(2\pi)^4} \left(2g_{ij} - \frac{g_{ij'}\tilde{G}_L^{i'j'}k_{i'}k_j}{\tilde{G}_L^{kk'}k_kk_{k'}} - \frac{g_{jj'}\tilde{G}_L^{i'j'}k_{i'}k_i}{\tilde{G}_L^{kk'}k_kk_{k'}}\right) \delta_{\vec{k},\vec{k}'} \\
&\quad - \left(\frac{R_1}{R_6\sqrt{g}}\right) \frac{1}{4(2\pi)^4} \frac{1}{\tilde{G}_L^{kk'}k_kk_{k'}} (g_{ii'}\gamma^{i'}k_j + g_{ji'}\gamma^{i'}k_i - \frac{2k_ik_j\gamma^lk_l}{g^{i'j'}k_{i'}k_{j'}}) \delta_{\vec{k},\vec{k}'}. \quad (3.46)
\end{aligned}$$

From the right side of (3.46) it can shown by explicit calculation that

$$\begin{aligned}
k^ik^j [a_{\vec{k}i}, a_{\vec{k}'j}^\dagger] &= 0, \quad g^{ij'}g^{j'i'}k_{i'}k_{j'} [a_{\vec{k}i}, a_{\vec{k}'j}^\dagger] = 0, \\
g^{ij'}g^{i'j'}k_{i'}k_{j'} [a_{\vec{k}i}, a_{\vec{k}'j}^\dagger] &= \left(\frac{R_1}{R_6\sqrt{g}}\right) R_6\sqrt{g^{ij}k_ik_j} \frac{3}{2(2\pi)^4}. \quad (3.47)
\end{aligned}$$

Since (3.47) is independent of R_6 , this expression is valid also for the Euclidean signature metric. The prefactors in the action (3.45) are those appropriate for the Euclidean signature metric, so

$$\begin{aligned}
I^{5d,Maxwell} &= \frac{(2\pi)^5 R_6\sqrt{g}}{R_1} \sum_{\vec{k} \neq 0} \left(k^ik^j + (g^{ij}g^{i'j'} - g^{ij'}g^{ji'})k_{i'}k_{j'}\right) (2a_{\vec{k}i}^\dagger a_{\vec{k}j} + [a_{\vec{k}i}, a_{\vec{k}j}^\dagger]) \\
&= \frac{(2\pi)^5 R_6\sqrt{g}}{R_1} \sum_{\vec{k} \neq 0} \left(k^ik^j + (g^{ij}g^{i'j'} - g^{ij'}g^{ji'})k_{i'}k_{j'}\right) 2a_{\vec{k}i}^\dagger a_{\vec{k}j} \\
&\quad + \pi \sum_{k \neq 0} 3R_6\sqrt{g^{ij}k_ik_j}. \quad (3.48)
\end{aligned}$$

From (3.46) we can express the commutator in terms of the polarization vectors f_i^κ ,

$$[a_{\vec{k}i}, a_{\vec{k}'j}^\dagger] = f_i^\kappa f_j^{\lambda*} [a_{\vec{k}}^\kappa, a_{\vec{k}'}^{\lambda\dagger}] = f_i^\kappa f_j^{\lambda*} \delta^{\kappa\lambda} \delta_{\vec{k},\vec{k}'}. \quad (3.49)$$

where we choose the normalization

$$[a_{\vec{k}}^\kappa, a_{\vec{k}'}^{\lambda\dagger}] = \delta^{\kappa\lambda} \delta_{\vec{k},\vec{k}'}. \quad (3.50)$$

From (3.49) we have

$$\begin{aligned}
f_i^\kappa f_j^{\lambda*} \delta^{\kappa\lambda} &= \left(\frac{R_1}{R_6\sqrt{g}}\right) \frac{R_6^2}{4R_6\sqrt{g^{ij}k_ik_j}} \frac{1}{(2\pi)^4} \left(2g_{ij} - \frac{g_{ij'}\tilde{G}_L^{i'j'}k_{i'}k_j}{\tilde{G}_L^{kk'}k_kk_{k'}} - \frac{g_{jj'}\tilde{G}_L^{i'j'}k_{i'}k_i}{\tilde{G}_L^{kk'}k_kk_{k'}}\right) \\
&\quad - \left(\frac{R_1}{R_6\sqrt{g}}\right) \frac{1}{4(2\pi)^4} \frac{1}{\tilde{G}_L^{kk'}k_kk_{k'}} (g_{ii'}\gamma^{i'}k_j + g_{ji'}\gamma^{i'}k_i - \frac{2k_ik_j\gamma^lk_l}{g^{i'j'}k_{i'}k_{j'}}). \quad (3.51)
\end{aligned}$$

Then the first term in (3.48) becomes

$$\begin{aligned}
& \frac{(2\pi)^5 R_6 \sqrt{g}}{R_1} \sum_{\vec{k} \neq 0} \left(k^i k^j + (g^{ij} g^{i'j'} - g^{ij'} g^{ji'}) k_{i'} k_{j'} \right) 2a_{\vec{k}i}^\dagger a_{\vec{k}j} \\
&= \frac{(2\pi)^5 R_6 \sqrt{g}}{R_1} \sum_{\vec{k} \neq 0} \left(k^i k^j + (g^{ij} g^{i'j'} - g^{ij'} g^{ji'}) k_{i'} k_{j'} \right) 2f_i^\kappa f_j^{\lambda*} a_{\vec{k}}^{\kappa\dagger} a_{\vec{k}}^\lambda \\
&= 2\pi \sum_{\vec{k} \neq 0} R_6 \sqrt{g^{ij} k_i k_j} \delta^{\kappa\lambda} a_{\vec{k}}^{\kappa\dagger} a_{\vec{k}}^\lambda = 2\pi \sum_{\vec{k} \neq 0} R_6 \sqrt{g^{ij} k_i k_j} a_{\vec{k}}^{\kappa\dagger} a_{\vec{k}}^\kappa,
\end{aligned} \tag{3.52}$$

where we used (3.47) and have assumed that

$$\left(k^i k^j + (g^{ij} g^{i'j'} - g^{ij'} g^{ji'}) k_{i'} k_{j'} \right) 2f_i^\kappa f_j^{\lambda*} \tag{3.53}$$

is proportional to $\delta^{\kappa\lambda}$. Then we can rewrite action (3.48) as

$$-I^{5d, Maxwell} = -2i\pi \sum_{\substack{k_i = \vec{k} \in \mathbb{Z}^4 \neq 0 \\ 1 \leq \kappa \leq 3}} (-iR_6 \sqrt{g^{ij} k_i k_j}) a_{\vec{k}}^{\kappa\dagger} a_{\vec{k}}^\kappa - i\pi \sum_{\substack{k_i = \vec{k} \in \mathbb{Z}^4 \neq 0 \\ 1 \leq \kappa \leq 3}} (-iR_6 \sqrt{g^{ij} k_i k_j}) \delta^{\kappa\kappa}. \tag{3.54}$$

4 Comparison of Oscillator Traces Z_{osc}^{5d} and Z_{osc}^{6d}

In order to compare the partition functions of the two theories, we first review the calculation for the 6d chiral field from [3] setting the angles between the circle and five-torus $\alpha, \beta^i = 0$. The oscillator trace is evaluated by rewriting (1.2) as

$$\begin{aligned}
-2\pi R_6 \mathcal{H} + i2\pi \gamma^i P_i &= \frac{i\pi}{12} \int_0^{2\pi} d^5 \theta H_{lrs} \epsilon^{lrsmn} H_{6mn} = \frac{i\pi}{2} \int_0^{2\pi} d^5 \theta \sqrt{-G} H^{6mn} H_{6mn} \\
&= -i\pi \int_0^{2\pi} d^5 \theta (\Pi^{mn} H_{6mn} + H_{6mn} \Pi^{mn})
\end{aligned} \tag{4.1}$$

where the definitions $H^{6mn} = \frac{1}{6\sqrt{-G}} \epsilon^{mnlrs} H_{lrs}$ and $H_{6mn} = \frac{1}{6\sqrt{-G} G^{66}} \epsilon_{mnlrs} H^{lrs}$ follow from the self-dual equation of motion (2.1). $\Pi^{mn}(\vec{\theta}, \theta^6)$, the field conjugate to $B_{mn}(\vec{\theta}, \theta^6)$ is defined from the Lagrangian for a general (non-self-dual) two-form $I_6 = \int d^6 \theta (-\frac{\sqrt{-G}}{24}) H_{LMN} H^{LMN}$, so $\Pi^{mn} = \frac{\delta I_6}{\delta \partial_6 B_{mn}} = -\frac{\sqrt{-G}}{4} H^{6mn}$. The commutation relations of the two-form and its conjugate field $\Pi^{mn}(\vec{\theta}, \theta^6)$ are

$$\begin{aligned}
[\Pi^{rs}(\vec{\theta}, \theta^6), B_{mn}(\vec{\theta}', \theta^6)] &= -i\delta^5(\vec{\theta} - \vec{\theta}') (\delta_m^r \delta_n^s - \delta_n^r \delta_m^s), \\
[\Pi^{rs}(\vec{\theta}, \theta^6), \Pi^{mn}(\vec{\theta}', \theta^6)] &= [B_{rs}(\vec{\theta}, \theta^6), B_{mn}(\vec{\theta}', \theta^6)] = 0.
\end{aligned}$$

From the Bianchi identity $\partial_{[L} H_{MNP]} = 0$ and the fact that (2.1) implies $\partial^L H_{LMN} = 0$, then a solution to (2.1) is given by a solution to the homogeneous equations $\partial^L \partial_L B_{MN} = 0$,

$\partial^L B_{LN} = 0$. These have a plane wave solution

$$B_{MN}(\vec{\theta}, \theta^6) = f_{MN}(p)e^{ip \cdot \theta} + (f_{MN}(p)e^{ip \cdot \theta})^*; \quad G^{LN} p_L p_N = 0; \quad p^L f_{LN} = 0; \quad (4.2)$$

and quantum tensor field expansion

$$B_{mn}(\vec{\theta}, \theta^6) = \text{zero modes} + \sum_{\vec{p}=p_l \in \mathbb{Z}^5 \neq 0} (f_{mn}^\kappa b_{\vec{p}}^\kappa e^{ip \cdot \theta} + f_{mn}^{\kappa*} b_{\vec{p}}^{\kappa\dagger} e^{-ip^* \cdot \theta}) \quad (4.3)$$

for the three physical polarizations of the 6d chiral two-form [3], $1 \leq \kappa \leq 3$. Because oscillators with different polarizations commute, each polarization can be treated separately and the result then cubed. Without the zero mode term,

$$B_{mn}(\vec{\theta}, \theta^6) = \sum_{\vec{p} \neq 0} (b_{\vec{p}mn} e^{ip \cdot \theta} + b_{\vec{p}mn}^\dagger e^{-ip^* \cdot \theta}), \quad (4.4)$$

for $b_{\vec{p}mn} = f_{mn}^1 b_{\vec{p}}^1$ for example, with a similar expansion for $\Pi^{mn}(\vec{\theta}, \theta^6)$ in terms of $c_{\vec{p}}^{6mn\dagger}$. From (4.2) the momentum p_6 is

$$p_6 = -\gamma^i p_i - iR_6 \sqrt{g^{ij} p_i p_j + \frac{p_1^2}{R_1^2}}. \quad (4.5)$$

For the gauge choice $B_{6n} = 0$, the exponent (4.1) becomes

$$\begin{aligned} & -i\pi(2\pi)^5 \sum_{\vec{p}=p_l \in \mathbb{Z}^5 \neq 0} ip_6 (\mathcal{C}_{\vec{p}}^{6mn\dagger} B_{\vec{p}mn} + B_{\vec{p}mn} \mathcal{C}_{\vec{p}}^{6mn\dagger}) \\ & = -2i\pi \sum_{\vec{p} \neq 0} p_6 \mathcal{C}_{\vec{p}}^{\kappa\dagger} B_{\vec{p}}^\lambda f^{\kappa mn}(p) f_{mn}^\lambda(p) - i\pi \sum_{\vec{p} \neq 0} p_6 f^{\kappa mn}(p) f_{mn}^\kappa(p) \\ & = -2i\pi \sum_{\vec{p} \neq 0} p_6 \mathcal{C}_{\vec{p}}^{\kappa\dagger} B_{\vec{p}}^\kappa - i\pi \sum_{\vec{p} \neq 0} p_6 \delta^{\kappa\kappa}, \end{aligned} \quad (4.6)$$

with $B_{\vec{p}mn} \equiv b_{\vec{p}mn} + b_{-\vec{p}mn}^\dagger$, $\mathcal{C}_{\vec{p}}^{6mn\dagger} \equiv c_{-\vec{p}}^{6mn} + c_{\vec{p}}^{6mn\dagger}$. The polarization tensors have been restored where $1 \leq \kappa, \lambda \leq 3$ and the oscillators $B_{\vec{p}}^\lambda, \mathcal{C}_{\vec{p}}^{\lambda\dagger}$ satisfy the commutation relation

$$[B_{\vec{p}}^\lambda, \mathcal{C}_{\vec{p}}^{\lambda\dagger}] = \delta^{\kappa\lambda} \delta_{\vec{p}, \vec{p}'}. \quad (4.7)$$

So restricting the manifold to a circle times a five-torus in [3] we have

$$\begin{aligned} & -2\pi R_6 \mathcal{H} + i2\pi \gamma^i P_i \\ & = -2i\pi \sum_{\vec{p} \in \mathbb{Z}^5 \neq 0} \left(-\gamma^i p_i - iR_6 \sqrt{g^{ij} p_i p_j + \frac{p_1^2}{R_1^2}} \right) \mathcal{C}_{\vec{p}}^{\kappa\dagger} B_{\vec{p}}^\kappa - \pi R_6 \sum_{\vec{p} \in \mathbb{Z}^5} \sqrt{g^{ij} p_i p_j + \frac{p_1^2}{R_1^2}} \delta^{\kappa\kappa} \end{aligned} \quad (4.8)$$

The oscillator trace (1.2) is

$$\begin{aligned}
Z_{\text{osc}}^{6d} &= \text{tr} e^{-t\mathcal{H}+i2\pi\gamma^i P_i} = \text{tr} e^{-2i\pi \sum_{\vec{p} \neq 0} p_6 C_{\vec{p}}^{\kappa^\dagger} B_{\vec{p}}^\kappa - \pi R_6 \sum_{\vec{p}} \sqrt{g^{ij} p_i p_j + \frac{p_1^2}{R_1^2}} \delta^{\kappa\kappa}}, \\
Z_{\text{chiral}}^{6d} &= Z_{\text{zero modes}}^{6d} \cdot \left(e^{-\pi R_6 \sum_{\vec{n} \in \mathcal{Z}^5} \sqrt{g^{ij} n_i n_j + \frac{n_1^2}{R_1^2}}} \prod_{\vec{n} \in \mathcal{Z}^5 \neq \vec{0}} \frac{1}{1 - e^{-2\pi R_6 \sqrt{g^{ij} n_i n_j + \frac{n_1^2}{R_1^2}} + i2\pi\gamma^i n_i}} \right)^3.
\end{aligned} \tag{4.9}$$

Regularizing the vacuum energy as in [3], the chiral field partition function (1.2) becomes

$$Z_{\text{chiral}}^{6d} = Z_{\text{zero modes}}^{6d} \cdot \left(e^{\frac{R_6 \pi^{-3} \sum_{\vec{n} \neq \vec{0}} \sqrt{G_5}}{(g_{ij} n^i n^j + R_1^2 (n^1)^2)^3}} \prod_{\vec{n} \in \mathcal{Z}^5 \neq \vec{0}} \frac{1}{1 - e^{-2\pi R_6 \sqrt{g^{ij} n_i n_j + \frac{(n_1)^2}{R_1^2}} + i2\pi\gamma^i n_i}} \right)^3. \tag{4.10}$$

Lastly we compute the 5d Maxwell partition function (1.1) from (3.54),

$$Z_{\text{Maxwell}}^{5d} = Z_{\text{zero modes}}^{5d} \cdot \text{tr} e^{-2i\pi \sum_{\vec{k} \neq \vec{0}} (-iR_6 \sqrt{g^{ij} k_i k_j}) a_{\vec{k}}^{\kappa^\dagger} a_{\vec{k}'}^\kappa - i\pi \sum_{\vec{k} \neq \vec{0}} (-iR_6 \sqrt{g^{ij} k_i k_j}) \delta^{\kappa\kappa}}, \tag{4.11}$$

where $\vec{k} = k_i = n_i \in \mathcal{Z}^4$ on the torus. From the standard Fock space argument

$$\text{tr} \omega^{\sum_p p a_p^\dagger a_p} = \prod_p \sum_{k=0}^{\infty} \langle k | \omega^{p a_p^\dagger a_p} | k \rangle = \prod_p \frac{1}{1 - \omega^p},$$

we perform the trace on the oscillators,

$$Z_{\text{osc}}^{5d} = \left(e^{-\pi R_6 \sum_{\vec{n} \in \mathcal{Z}^4} \sqrt{g^{ij} n_i n_j}} \prod_{\vec{n} \in \mathcal{Z}^4 \neq \vec{0}} \frac{1}{1 - e^{-i2\pi(-iR_6 \sqrt{g^{ij} n_i n_j})}} \right)^3, \tag{4.12}$$

$$Z_{\text{Maxwell}}^{5d} = Z_{\text{zero modes}}^{5d} \cdot \left(e^{-\pi R_6 \sum_{\vec{n} \in \mathcal{Z}^4} \sqrt{g^{ij} n_i n_j}} \prod_{\vec{n} \in \mathcal{Z}^4 \neq \vec{0}} \frac{1}{1 - e^{-2\pi R_6 \sqrt{g^{ij} n_i n_j}}} \right)^3. \tag{4.13}$$

(4.13) is manifestly $SL(4, \mathcal{Z})$ invariant due to the underlying $SO(4)$ invariance we have labeled as $i = 2, 3, 4, 5$. $Z_{\text{zero modes}}^{5d}$ given in (2.12) is equivalent to the zero mode sum of the six-dimensional chiral theory restricted to $T^5 \times S^1$, up to an $SL(4, \mathcal{Z})$ -invariant constant factor (1.3), so $Z_{\text{zero modes}}^{5d}$ has $SL(4, \mathcal{Z})$ invariance. We use the $SL(4, \mathcal{Z})$ invariant regularization of the vacuum energy reviewed in Appendix C to obtain

$$Z_{\text{Maxwell}}^{5d} = Z_{\text{zero modes}}^{5d} \cdot \left(e^{\frac{\frac{3}{8} R_6 \pi^{-2} \sum_{\vec{n} \neq \vec{0}} \frac{\sqrt{g}}{(g_{ij} n^i n^j)^{\frac{5}{2}}}}}{\prod_{\vec{n} \in \mathcal{Z}^4 \neq \vec{0}} \frac{1}{1 - e^{-2\pi R_6 \sqrt{g^{ij} n_i n_j}}}} \right)^3, \tag{4.14}$$

where the sum is on the original lattice $\vec{n} = n^i \in \mathcal{Z}^4 \neq \vec{0}$, and the product is on the dual lattice $\vec{n} = n_i \in \mathcal{Z}^4 \neq \vec{0}$. Of course the product of the zero mode contribution and the oscillator contribution in (4.13) has $SL(5, \mathcal{Z})$ invariance by construction from (1.1).

The comparison of the 6d chiral theory on $S^1 \times T^5$ and the abelian gauge theory on T^5 shows the exponent of the oscillator contribution to the partition function for the 6d theory (4.8),

$$\begin{aligned}
& -2\pi R_6 \mathcal{H} + i2\pi \gamma^i P_i \\
& = -2\pi \sum_{\vec{p} \in \mathcal{Z}^5 \neq 0} \left(-i\gamma^i p_i + R_6 \sqrt{g^{ij} p_i p_j + \frac{p_1^2}{R_1^2}} \right) \mathcal{C}_{\vec{p}}^{\kappa\dagger} B_{\vec{p}}^{\kappa} - \pi R_6 \sum_{\vec{p} \in \mathcal{Z}^5} \sqrt{g^{ij} p_i p_j + \frac{p_1^2}{R_1^2}} \delta^{\kappa\kappa},
\end{aligned} \tag{4.15}$$

and for the gauge theory (3.54),

$$-I^{5d, Maxwell} = -2\pi \sum_{\vec{k} \in \mathcal{Z}^4 \neq 0} \left(R_6 \sqrt{g^{ij} k_i k_j} \right) a_{\vec{k}}^{\kappa\dagger} a_{\vec{k}}^{\kappa} - \pi R_6 \sum_{\vec{k} \in \mathcal{Z}^4} \sqrt{g^{ij} k_i k_j} \delta^{\kappa\kappa}, \tag{4.16}$$

differ by the sum on the Kaluza-Klein modes p_1 of S^1 since for the chiral case $\vec{p} \in \mathcal{Z}^5$, and for the Maxwell case $\vec{k} \in \mathcal{Z}^4$. More suprising is the different dependence on $\gamma^i \sim \tilde{G}_5^{i6}$, the angle between the space directions θ^i and the time direction θ^6 , although we would not expect to see this phase in the Maxwell action because it is real. We note that the modified Hamiltonian U for the Maxwell theory discussed in Appendix B supplies a term somewhat similar to $\gamma^i P_i$, but using U to compute the partition function instead of $H \sim L_E$ does not provide a bounded zero mode sum.

Both theories have three polarizations, $1 \leq \kappa \leq 3$, and from (4.7) and (3.50) the oscillators have the same commutation relations,

$$[B_{\vec{p}}^{\lambda}, \mathcal{C}_{\vec{p}}^{\lambda\dagger}] = \delta^{\kappa\lambda} \delta_{\vec{p}, \vec{p}'}, \quad [a_{\vec{k}}^{\kappa}, a_{\vec{k}'}^{\lambda\dagger}] = \delta^{\kappa\lambda} \delta_{\vec{k}, \vec{k}'}. \tag{4.17}$$

If we neglect the phase $(-i\gamma^i p_i)$, and discard the Kaluza-Klein modes p_1^2 in the usual limit [25] as the radius of the circle R_1 is very small with respect to the radii and angles g_{ij}, R_6 , of the five-torus, then the oscillator products in (4.10) and (4.14) are equivalent. This holds since we can separate the product on $\vec{n} = (n_1, n_i) \neq \vec{0}$ into a product on $(n_1 = 0, \text{all } n_i \neq (0, 0, 0, 0))$ and on $(n_1 \neq 0, \text{all } n_i)$, to find

$$\prod_{\vec{n} \in \mathcal{Z}^5 \neq \vec{0}} \frac{1}{1 - e^{-2\pi R_6 \sqrt{g^{ij} n_i n_j + \frac{(n_1)^2}{R_1^2}}}} = \prod_{n_i \neq (0, 0, 0, 0)} \frac{1}{1 - e^{-2\pi R_6 \sqrt{g^{ij} n_i n_j}}} \cdot \prod_{n_1 \neq 0, n_i \in \mathcal{Z}^4} \frac{1}{1 - e^{-2\pi R_6 \sqrt{g^{ij} n_i n_j + \frac{(n_1)^2}{R_1^2}}}} \tag{4.18}$$

In the limit of small R_1 the last product reduces to unity, thus for S^1 smaller than T^5

$$\prod_{\vec{n} \in \mathcal{Z}^5 \neq \vec{0}} \frac{1}{1 - e^{-2\pi R_6 \sqrt{g^{ij} n_i n_j + \frac{(n_1)^2}{R_1^2}}}} \rightarrow \prod_{n_i \in \mathcal{Z}^4 \neq \vec{0}} \frac{1}{1 - e^{-2\pi R_6 \sqrt{g^{ij} n_i n_j}}}. \tag{4.19}$$

This thus holds for the oscillator contribution to the partition functions (4.9) and (4.13), at least up to regularizing the vacuum energy which involves interchanging a sum and a limit;

and together with (1.3) in this limit we have

$$Z^{6d,chiral} \rightarrow Z^{5d,Maxwell}. \quad (4.20)$$

5 Discussion and Conclusions

The Maxwell field theory is quantized on a twisted torus T^5 with the Dirac method of constraints, resulting in the commutation relations in (3.30). Although these are complicated, they lead to a simple expression for the Euclidean action (3.54). The partition function is then found as a product of zero mode and oscillator contributions. Comparing this with the partition function of a 6d self-dual three-form field strength [3] restricted here to $S^1 \times T^5$ where the radius of the circle is $R_1 \equiv g_{5YM}^2$, we compute it to be equivalent in the limit where R_1 is small relative to the five-torus, up to an overall scale relating the zero modes (1.3), (2.12) and a phase occurring in the oscillator sums (4.9), (4.13), by removing the Kaluza-Klein modes from the 6d partition sum. How to incorporate these modes rigorously in a partition function as instantons in the non-abelian version of the 5d Maxwell theory with appropriate dynamics remains difficult [28]-[30], suggesting that the 6d finite conformal $N = (2, 0)$ theory on a circle is an ultraviolet completion of the 5d maximally supersymmetric Maxwell theory rather than an exact quantum equivalence.

Here we present a comparison of the partition functions of the two theories, for the abelian case without supersymmetry, when the five-dimensional manifold is a torus and the angles between the five-torus and the circle are zero.

It would be compelling to find how expressions for the partition function of the 6d $N = (2, 0)$ conformal quantum theory computed on various manifolds using localization should reduce to the expression in [3] in an appropriate limit, providing a check that localization is equivalent to Dirac quantization.

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A Dirac Method of Quantization with Constraints

The 5d Maxwell theory on a five-torus with general flat metric has the Hamiltonian (3.16),

$$H_p = \int d^4\theta \left(-\frac{g_{ii'}}{2} \Pi^i \Pi^{i'} + \frac{\tilde{G}_L^{66}}{4} g^{ii'} g^{jj'} F_{ij} F_{i'j'} - \tilde{G}_L^{66} \partial_i \Pi^i A_6 - \tilde{G}_L^{6j} \partial_i \Pi^i A_j + \lambda_1 \Pi^6 \right) \quad (\text{A.1})$$

with λ_1 as a Lagrange multiplier. To quantize and derive the commutation relations, we start with the equal-time *canonical Poisson brackets*

$$\begin{aligned} \{\Pi^{\vec{m}}(\vec{\theta}, \theta^6), A_{\vec{n}}(\vec{\theta}', \theta^6)\} &= -\{A_{\vec{n}}(\vec{\theta}', \theta^6), \Pi^{\vec{m}}(\vec{\theta}, \theta^6)\} = -\delta^4(\vec{\theta} - \vec{\theta}') \delta_{\vec{n}}^{\vec{m}}, \\ \{\Pi^{\vec{m}}(\vec{\theta}, \theta^6), \Pi^{\vec{n}}(\vec{\theta}', \theta^6)\} &= \{A_{\vec{m}}(\vec{\theta}, \theta^6), A_{\vec{n}}(\vec{\theta}', \theta^6)\} = 0. \end{aligned} \quad (\text{A.2})$$

The constraints are required to be time-independent, so from (3.11) for $\phi^1(\theta) \equiv \Pi^6(\vec{\theta}, \theta^6)$,

$$-\partial_6 \phi^1(\vec{\theta}, \theta^6) = \{\phi^1(\vec{\theta}, \theta^6), H_p\} = -\tilde{G}_L^{66} \int d^4\theta' \{\Pi^6(\theta), A_6(\theta')\} \partial_i \Pi^i(\theta') = \tilde{G}_L^{66} \partial_i \Pi^i(\theta) \approx 0. \quad (\text{A.3})$$

Thus the secondary constraint is

$$\phi^2(\theta) = \partial_i \Pi^i(\vec{\theta}, \theta^6) \approx 0. \quad (\text{A.4})$$

The secondary constraint is time-independent from two contributions:

$$\begin{aligned} -\partial_6 \phi^2(\vec{\theta}, \theta^6) &= \{\phi^2(\vec{\theta}, \theta^6), H_p^{(1)}\} = \frac{\tilde{G}_L^{66}}{4} g^{ii'} g^{jj'} \int d^4\theta' \{\partial_k \Pi^k(\theta), F_{ij}(\theta') F_{i'j'}(\theta')\} = 0, \\ -\partial_6 \phi^2(\vec{\theta}, \theta^6) &= \{\phi^2(\vec{\theta}, \theta^6), H_p^{(2)}\} = -\tilde{G}_L^{6j} \int d^4\theta' \{\partial_k \Pi^k(\theta), \partial_i \Pi^i(\theta') A_j(\theta')\} = \tilde{G}_L^{6j} \partial_j \partial_i \Pi^i(\theta) \approx 0. \end{aligned} \quad (\text{A.5})$$

The two constraints ϕ^1, ϕ^2 are first class constraints since they have vanishing Poisson bracket,

$$\{\Pi^6(\theta), \partial_i \Pi^i(\theta')\} = 0. \quad (\text{A.6})$$

We introduce the gauge conditions

$$\chi^1(\theta) = A_6(\theta) \approx 0, \quad \chi^2(\theta) = \partial^i A_i(\theta) = G_L^{ij} \partial_j A_i - \Pi^i g_{ij} \gamma^j + \tilde{G}_L^{i6} \partial_i A_6 \approx 0. \quad (\text{A.7})$$

These convert all four constraints to second class, *i.e.* all now have at least one non-vanishing Poisson bracket with each other, where the non-vanishing brackets are

$$\begin{aligned}
\{\phi^1(\theta), \chi^1(\theta')\} &= \{\Pi^6(\theta), A_6(\theta')\} = -\delta^4(\theta - \theta') = -\{A_6(\theta), \Pi^6(\theta')\}, \\
\{\phi^1(\theta), \chi^2(\theta')\} &= \{\Pi^6(\theta), \tilde{G}_L^{i6} \partial_i A_6(\theta')\} = \tilde{G}_L^{i6} \frac{\partial}{\partial \theta^i} \delta^4(\theta - \theta') = \{\tilde{G}_L^{i6} \partial_i A_6(\theta), \Pi^6(\theta')\}, \\
\{\phi^2(\theta), \chi^2(\theta')\} &= \{\partial_i \Pi^i(\theta), \tilde{G}_L^{jj'} \partial_{j'} A_j(\theta')\} = \tilde{G}_L^{ij} \frac{\partial}{\partial \theta^i} \frac{\partial}{\partial \theta^j} \delta^4(\theta - \theta') = -\{\tilde{G}_L^{jj'} \partial_{j'} A_j(\theta), \partial_i \Pi^i(\theta')\}.
\end{aligned} \tag{A.8}$$

We can write these as a matrix for the four constraints $\phi^A = (\phi^a, \chi^a)$ as $C^{AB}(\theta, \theta') \equiv \{\phi^A(\theta), \phi^B(\theta')\}$,

$$C^{AB} = \begin{pmatrix} 0 & 0 & -1 & \tilde{G}_L^{i6} \frac{\partial}{\partial \theta^i} \\ 0 & 0 & 0 & \tilde{G}_L^{ij} \frac{\partial}{\partial \theta^i} \frac{\partial}{\partial \theta^j} \\ 1 & 0 & 0 & 0 \\ \tilde{G}_L^{i6} \frac{\partial}{\partial \theta^i} & -\tilde{G}_L^{ij} \frac{\partial}{\partial \theta^i} \frac{\partial}{\partial \theta^j} & 0 & 0 \end{pmatrix} \delta^4(\theta - \theta'). \tag{A.9}$$

The inverse matrix is

$$(C_{AB})^{-1} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{\tilde{G}_L^{i6} \frac{\partial}{\partial \theta^i}}{\tilde{G}_L^{kk'} \frac{\partial}{\partial \theta^k} \frac{\partial}{\partial \theta^{k'}}} & -\frac{1}{\tilde{G}_L^{kk'} \frac{\partial}{\partial \theta^k} \frac{\partial}{\partial \theta^{k'}}} \\ -1 & \frac{\tilde{G}_L^{i6} \frac{\partial}{\partial \theta^i}}{\tilde{G}_L^{kk'} \frac{\partial}{\partial \theta^k} \frac{\partial}{\partial \theta^{k'}}} & 0 & 0 \\ 0 & \frac{1}{\tilde{G}_L^{kk'} \frac{\partial}{\partial \theta^k} \frac{\partial}{\partial \theta^{k'}}} & 0 & 0 \end{pmatrix} \delta^4(\theta - \theta'). \tag{A.10}$$

The Dirac bracket is defined to vanish with any constraint,

$$\begin{aligned}
\{A_{\tilde{m}}(\theta), \Pi^{\tilde{n}}(\theta')\}_D &= \{A_{\tilde{m}}(\theta), \Pi^{\tilde{n}}(\theta')\} - \int d^4\rho d^4\rho' \Big(\{A_{\tilde{m}}(\theta), \Pi^6(\rho)\} C_{13}^{-1} \{A_6(\rho'), \pi^{\tilde{n}}(\theta')\} \\
&\quad + \{A_{\tilde{m}}(\theta), \partial_i \Pi^i(\rho)\} C_{23}^{-1} \{A_6(\rho'), \pi^{\tilde{n}}(\theta')\} \\
&\quad + \{A_{\tilde{m}}(\theta), \partial_i \Pi^i(\rho)\} C_{24}^{-1} \{\partial^j A_j(\rho'), \Pi^{\tilde{n}}(\theta')\} \\
&\quad + \{A_{\tilde{m}}(\theta), A_6(\rho)\} C_{31}^{-1} \{\Pi^6(\rho'), \pi^{\tilde{n}}(\theta')\} \\
&\quad + \{A_{\tilde{m}}(\theta), A_6(\rho)\} C_{32}^{-1} \{\partial_i \Pi^i(\rho'), \Pi^{\tilde{n}}(\theta')\} \\
&\quad + \{A_{\tilde{m}}(\theta), \partial^j A_j(\rho)\} C_{42}^{-1} \{\partial_i \Pi^i(\rho'), \Pi^{\tilde{n}}(\theta')\}. \Big)
\end{aligned} \tag{A.11}$$

So

$$\begin{aligned}\{A_i(\theta), \Pi^j(\theta')\}_D &= \{A_i(\theta), \pi^j(\theta')\} - \int d^4\rho d^4\rho' \left(\{A_i(\theta), \partial_k \Pi^k(\rho)\} C_{24}^{-1} \{\partial^{k'} A_{k'}(\rho'), \pi^j(\theta')\} \right) \\ &= \left(\delta_i^j - \tilde{G}_L^{jj'} (\partial_i \frac{1}{\tilde{G}_L^{kk'} \partial_k \partial_{k'}} \partial_{j'}) \right) \delta^4(\theta - \theta'),\end{aligned}\tag{A.12}$$

where here all ∂_j are with respect to θ^j . So promoting the Dirac Poisson bracket to a quantum commutator, we derive the equal time commutation relations

$$[\Pi^j(\vec{\theta}, \theta^6), A_i(\vec{\theta}', \theta^6)] = -i \left(\delta_i^j - \tilde{G}_L^{jj'} (\partial_i \frac{1}{\tilde{G}_L^{kk'} \partial_k \partial_{k'}} \partial_{j'}) \right) \delta^4(\theta - \theta'),\tag{A.13}$$

and similarly,

$$[A_i(\vec{\theta}, \theta^6), A_j(\vec{\theta}', \theta^6)] = 0, \quad [\Pi^i(\vec{\theta}, \theta^6), \Pi^j(\vec{\theta}', \theta^6)] = 0.\tag{A.14}$$

B Modified Hamiltonian and Equations of Motion

We need to check that the Hamiltonian gives the correct equations of motion for $A_6 = 0$ which are derived from L given in (3.8):

$$\begin{aligned}\partial^{\tilde{m}} F_{\tilde{m}\tilde{n}} &= \partial^{\tilde{m}} \partial_{\tilde{m}} A_{\tilde{n}} - \partial_{\tilde{n}} \partial^{\tilde{m}} A_{\tilde{m}} \\ &\Rightarrow \tilde{G}_L^{ij} \partial_i \partial_j A_k + 2\tilde{G}_L^{i6} \partial_i \partial_6 A_k + \tilde{G}_L^{66} \partial_6 \partial_6 A_k - \partial_k \tilde{G}_L^{ij} \partial_j A_i - \partial_k \tilde{G}_L^{i6} \partial_6 A_i = 0, \quad \text{for } \tilde{n} = k\end{aligned}\tag{B.1}$$

$$\Rightarrow \tilde{G}_L^{k6} \partial_6 \partial_6 A_k + \tilde{G}_L^{ki'} \partial_{i'} \partial_6 A_k = 0, \quad \text{for } \tilde{n} = 6\tag{B.2}$$

Following Dirac [26], the procedure to get the relevant Hamiltonian H_E for checking the equations of motion is to set $\Pi^6 = 0$ in (3.15) and add on the first-class secondary constraint $\phi^2 = \partial_i \Pi^i$ with an arbitrary coefficient $u(\theta)$. Doing this we do not get the γ^i terms in the equations of motion (B.1). In order to correct for this and find the equations of motion, we must add $\tilde{G}_L^{66} \gamma^j F_{ij} \Pi^i$ to our original Lorentzian Hamilton (3.9). So we will use

$$\begin{aligned}H_E &= \int d^4\theta \left(-\frac{g_{ii'}}{2} \Pi^i \Pi^{i'} + \frac{\tilde{G}_L^{66}}{4} g^{ii'} g^{jj'} F_{ij} F_{i'j'} - \tilde{G}_L^{66} \partial_i \Pi^i A_6 - \tilde{G}_L^{6j} \partial_i \Pi^i A_j + \tilde{G}_L^{66} \gamma^j F_{ij} \Pi^i + u \partial_i \Pi^i \right) \\ &= \int d^4\theta \left(-\frac{g_{ii'}}{2} \Pi^i \Pi^{i'} + \frac{\tilde{G}_L^{66}}{4} g^{ii'} g^{jj'} F_{ij} F_{i'j'} - \tilde{G}_L^{66} \gamma^j \partial_j A_i \Pi^i + u \partial_i \Pi^i \right),\end{aligned}\tag{B.3}$$

where we integrated by parts and absorbed $-\tilde{G}_L^{66} A_6 - 2\tilde{G}_L^{6j} A_j$ by shifting the arbitrary coefficient u . Then the Hamilton equations of motion are

$$\begin{aligned}
-\partial_6 \Pi^k(\theta) &= \{\Pi^k(\theta), H_E\} = \tilde{G}_L^{66}(g^{ii'}g^{kj'} - g^{ij'}g^{ki'})\partial_i\partial_{i'}A_{j'}(\theta) - \tilde{G}_L^{66}\gamma^j\partial_j\Pi^k(\theta), \\
-\partial_6 A_k(\theta) &= \{A_k(\theta), H_E\} = -g_{ki}\Pi^i(\theta) - \partial_k u(\theta) - \tilde{G}_L^{66}\gamma^j\partial_j A_k(\theta),
\end{aligned} \tag{B.4}$$

where regular Poisson brackets are used to compute the time evolution as in (A.5). At this point it is necessary to reinsert the dimensionful parameters, which will make H'_E and Π'^k dimensionless along with A_k , and use a ‘primed’ Poisson bracket defined by variation with respect to the dimensionless fields,

$$H_E = \frac{R_1}{R_6^3\sqrt{g}} H'_E, \quad \Pi^k(\theta) = \frac{R_1}{R_6\sqrt{g}} \Pi'^k(\theta), \tag{B.5}$$

$$-\partial_6 \Pi'^k(\theta) = \{\Pi'^k(\theta), H'_E\}' = -\frac{R_6\sqrt{g}}{R_1}(g^{ii'}g^{kj'} - g^{ij'}g^{ki'})\partial_i\partial_{i'}A_{j'}(\theta) + \gamma^j\partial_j\Pi'^k(\theta), \tag{B.6}$$

$$-\partial_6 A_k(\theta) = \{A_k(\theta), H'_E\}' = -\frac{R_1 R_6}{\sqrt{g}} g_{ki}\Pi'^i(\theta) - \partial_k u(\theta) + \gamma^j\partial_j A_k(\theta). \tag{B.7}$$

Faraday’s Law

Using (B.7), we have

$$\begin{aligned}
-\partial_6 \partial_6 A_k &= -\frac{R_1 R_6}{\sqrt{g}} g_{kk'} \partial_6 \Pi'^{k'} - \partial_6 \partial_k u + \gamma^j \partial_6 \partial_j A_k \\
&= -R_6^2 g^{ii'} \partial_i \partial_{i'} A_k + R_6^2 g^{ij'} \partial_i \partial_k A_{j'} + \frac{R_1 R_6}{\sqrt{g}} g_{kk'} \gamma^j \partial_j \Pi'^{k'} - \partial_6 \partial_k u + \gamma^j \partial_6 \partial_j A_k.
\end{aligned} \tag{B.8}$$

Reexpressing the conjugate momentum $\Pi'^{k'}(\theta)$ in the terms of the velocities via (3.10),

$$\Pi'^k = \frac{\sqrt{g}}{R_1 R_6} g^{ki} (\partial_6 A_i + \gamma^j \partial_j A_i - \gamma^j \partial_i A_j), \tag{B.9}$$

we find the equations of motion computed from (B.8) to be

$$\begin{aligned}
0 &= \partial_6 \partial_6 A_k - R_6^2 g^{ii'} \partial_i \partial_{i'} A_k + R_6^2 g^{ij'} \partial_i \partial_k A_{j'} + \frac{R_1 R_6}{\sqrt{g}} g_{kk'} \gamma^j \partial_j \Pi'^{k'} - \partial_6 \partial_k u + \gamma^j \partial_6 \partial_j A_k \\
&= \partial_6 \partial_6 A_k - R_6^2 g^{ii'} \partial_i \partial_{i'} A_k + R_6^2 g^{ij'} \partial_i \partial_k A_{j'} + 2\gamma^j \partial_j \partial_6 A_k + \gamma^j \gamma^{j'} \partial_j \partial_{j'} A_k - \gamma^j \gamma^{j'} \partial_{j'} \partial_k A_j - \partial_6 \partial_k u
\end{aligned} \tag{B.10}$$

which can be rewritten as

$$\tilde{G}_L^{66} \partial_6 \partial_6 A_k + \tilde{G}_L^{ii'} \partial_i \partial_{i'} A_k - \tilde{G}_L^{ij'} \partial_i \partial_k A_{j'} + 2\tilde{G}_L^{j6} \partial_j \partial_6 A_k + \partial_6 \partial_k \tilde{u} = 0, \tag{B.11}$$

where \tilde{u} has been rescaled and is arbitrary, and allows for a general gauge choice in this

equation. By choosing

$$\tilde{u} = -\tilde{G}_L^{j6} A_j = \frac{u}{R_6^2} \quad (\text{B.12})$$

we see that (B.11) gives the equations of motion (B.1) which are in $A_6 = 0$ gauge.

Gauss' Law

Checking the second equation of motion, we have from (B.10)

$$\begin{aligned} 0 &= -\frac{\gamma^k}{R_6^2} \partial_6 \partial_6 A_k + \gamma^k g^{ii'} \partial_i \partial_{i'} A_k - \gamma^k g^{ij'} \partial_i \partial_k A_{j'} - \frac{2}{R_6^2} \gamma^k \gamma^j \partial_j \partial_6 A_k + \frac{\gamma^k}{R_6^2} \partial_6 \partial_k u \\ &= -\frac{\gamma^k}{R_6^2} \partial_6 \partial_6 A_k + \partial_i g^{ii'} (\gamma^k \partial_{i'} A_k - \gamma^k \partial_k A_{i'} - \partial_6 A_{i'}) + \partial_i g^{ii'} \partial_6 A_{i'} - \frac{2}{R_6^2} \gamma^k \gamma^j \partial_j \partial_6 A_k + \frac{\gamma^k}{R_6^2} \partial_6 \partial_k u \\ &= \tilde{G}_L^{ik} \partial_6 \partial_6 A_k + \tilde{G}_L^{ii'} \partial_i \partial_6 A_{i'} - \frac{1}{R_6^2} \gamma^k \gamma^j \partial_j \partial_6 A_k + \frac{\gamma^k}{R_6^2} \partial_6 \partial_k u \end{aligned} \quad (\text{B.13})$$

where we used the constraint $\partial_i \Pi^i = 0$ and (B.9). Again choosing the gauge function u as in (B.12), we find that (B.13) reduces to the equations of motion (B.2) which are in $A_6 = 0$ gauge.

Dirac method of quantization with constraints using the modified Hamiltonian

We will now repeat our Dirac quantization using the modified Lorentzian Hamiltonian

$$\begin{aligned} U &\equiv H + \gamma^j P_j = \Pi^i \partial^6 A_i - \tilde{G}_L^{66} L + \tilde{G}_L^{66} \gamma^j F_{ij} \Pi^i \\ &= \Pi^i \tilde{G}_L^{66} F_{6i} + \Pi^i \tilde{G}_L^{6j} \partial_j A_i + \Pi^i \tilde{G}_L^{66} \partial_i A_6 - \tilde{G}_L^{66} L + \tilde{G}_L^{66} \gamma^j F_{ij} \Pi^i \\ &= \tilde{G}_L^{66} (-L_E) - \tilde{G}_L^{66} \gamma^j \partial_j A_i \Pi^i + \tilde{G}_L^{66} (2\gamma^j \Pi^i \partial_i A_j + \Pi^i \partial_i A_6) \end{aligned} \quad (\text{B.14})$$

instead of (3.9). Here

$$\gamma^j P_j \equiv \tilde{G}_L^{66} \gamma^j F_{ij} \Pi^i \quad (\text{B.15})$$

and U is independent of time since

$$\{\int d^4\theta \tilde{G}_L^{66} \gamma^j F_{ij}(\theta) \Pi^i(\theta), H_p\} = 0. \quad (\text{B.16})$$

Then instead of (3.16), we have the modified primary Hamiltonian

$$U_p = \int d^4\theta \left(-\frac{g^{ii'}}{2} \Pi^i \Pi^{i'} + \frac{\tilde{G}_L^{66}}{4} g^{ii'} g^{jj'} F_{ij} F_{i'j'} - \tilde{G}_L^{66} \gamma^j \partial_j A_i \Pi^i - \tilde{G}_L^{66} \partial_i \Pi^i A_6 - 2\tilde{G}_L^{6j} \partial_i \Pi^i A_j + \lambda_1 \Pi^6 \right), \quad (\text{B.17})$$

The first constraint $\phi^1(\theta) = \Pi^6(\vec{\theta}, \theta^6)$ is required to be independent of time:

$$-\partial_6 \phi^1(\vec{\theta}, \theta^6) = \{\phi^1(\vec{\theta}, \theta^6), U_p\} = -\tilde{G}_L^{66} \int d^4 \theta' \{\Pi^6(\theta), A_6(\theta')\} \partial_i \Pi^i(\theta') = \tilde{G}_L^{66} \partial_i \Pi^i(\theta) \approx 0. \quad (\text{B.18})$$

The secondary constraint is

$$\phi^2(\theta) = \partial_i \Pi^i(\vec{\theta}, \theta^6) \approx 0. \quad (\text{B.19})$$

Its time derivative has three contributions

$$\begin{aligned} -\partial_6 \phi^2(\vec{\theta}, \theta^6) &= \{\phi^2(\vec{\theta}, \theta^6), U_p^{(1)}\} = \frac{\tilde{G}_L^{66}}{4} g^{ii'} g^{jj'} \int d^4 \theta' \{\partial_k \Pi^k(\theta), F_{ij}(\theta') F_{i'j'}(\theta')\} \\ &= \frac{\tilde{G}_L^{66}}{2} (g^{ii'} g^{jj'} - g^{ij'} g^{ji'}) \int d^4 \theta' \{\partial_k \Pi^k(\theta), \partial_i A_j(\theta') \partial_{i'} A_{j'}(\theta')\} = \tilde{G}_L^{66} (g^{ii'} g^{jj'} - g^{ij'} g^{ji'}) \partial_j \partial_i \partial_{i'} A_{j'}(\theta) = 0. \\ -\partial_6 \phi^2(\vec{\theta}, \theta^6) &= \{\phi^2(\vec{\theta}, \theta^6), U_p^{(2)}\} = -2\tilde{G}_L^{6j} \int d^4 \theta' \{\partial_k \Pi^k(\theta), \partial_i \Pi^i(\theta') A_j(\theta')\} = 2\tilde{G}_L^{6j} \partial_j \partial_i \Pi^i(\theta) \approx 0. \\ -\partial_6 \phi^2(\vec{\theta}, \theta^6) &= \{\phi^2(\vec{\theta}, \theta^6), U_p^{(3)}\} = -\tilde{G}_L^{66} \gamma^j \int d^4 \theta' \{\partial_k \Pi^k(\theta), \partial_j A_i(\theta') \Pi^i(\theta')\} = \tilde{G}_L^{6j} \partial_j \partial_i \Pi^i(\theta) \approx 0. \end{aligned} \quad (\text{B.20})$$

Thus the constraints are the same as for the original Hamiltonian (A.1), and hence the equal time commutators (A.13) and (A.14) are unchanged.

C Regularization of the Vacuum Energy for 5d Maxwell Theory

The Fourier transform of powers of a radial function is

$$|\vec{p}|^{\alpha-n} = \frac{c_\alpha}{(2\pi)^n} \int d^n y \sqrt{G_n} e^{-i\vec{p} \cdot \vec{y}} \frac{1}{|\vec{y}|^\alpha}, \quad \text{where} \quad c_\alpha \equiv \frac{\pi^{\frac{n}{2}} 2^\alpha \Gamma(\frac{\alpha}{2})}{\Gamma(\frac{n-\alpha}{2})}. \quad (\text{C.1})$$

This formula holds by analytic continuation, since for general n, α , where the area of the unit sphere S_{n-2} is

$$\omega_{n-2} = \frac{2\pi^{\frac{n-1}{2}}}{\Gamma(\frac{n-1}{2})} \equiv \int_0^\pi d\theta_1 d\theta_2 \dots d\theta_{n-3} \sin \theta_1 \sin^2 \theta_2 \dots \sin^{n-3} \theta_{n-3} \int_0^{2\pi} d\phi, \quad (\text{C.2})$$

the Fourier integral is

$$\begin{aligned}
\int d^n y \sqrt{G_n} e^{-i\vec{p}\cdot\vec{y}} \frac{1}{|\vec{y}|^\alpha} &= \int_0^\infty dy y^{n-1-\alpha} \int_0^\pi d\theta \sin^{n-2} \theta e^{-i|\vec{p}|y \cos \theta} \omega_{n-2} \\
&= \int_0^\infty dy y^{n-1-\alpha} \frac{(2\pi)^{\frac{n}{2}}}{(|\vec{p}|y)^{\frac{n-2}{2}}} J_{\frac{n-2}{2}}(|\vec{p}|y) \\
&= |\vec{p}|^{\alpha-n} (2\pi)^{\frac{n}{2}} \frac{2^{\frac{n}{2}-\alpha} \Gamma(\frac{n-\alpha}{2})}{\Gamma(\frac{\alpha}{2})},
\end{aligned} \tag{C.3}$$

where the last expression is valid for the integral when $-\frac{n}{2} < \frac{n}{2} - \alpha < \frac{1}{2}$, but can be analytically continued for all $\alpha \neq -n, -n-1, \dots$

So expressing $|\vec{p}|$ in terms of its 4d Fourier transform,

$$\begin{aligned}
|\vec{p}| &= -\frac{3}{4\pi^2} \int d^4 y \sqrt{g} e^{-i\vec{p}\cdot\vec{y}} \frac{1}{|\vec{y}|^5}, \\
\langle H \rangle &= \frac{1}{2} \sum_{\vec{p} \in \mathcal{Z}^4} |\vec{p}| e^{i\vec{p}\cdot\vec{x}}|_{\vec{x}=0} = \frac{1}{2} \sum_{\vec{k} \in \mathcal{Z}^4} \sqrt{g^{ij} p_i p_j},
\end{aligned} \tag{C.4}$$

we have

$$\begin{aligned}
\sum_{\vec{p} \in \mathcal{Z}^4} |\vec{p}| e^{i\vec{p}\cdot\vec{x}} &= -\frac{3}{4\pi^2} \sqrt{g} \int d^4 y \frac{1}{|\vec{y}|^5} \sum_{\vec{p}} e^{i\vec{p}\cdot(\vec{x}-\vec{y})} \\
&= -\frac{3}{4\pi^2} \sqrt{g} \int d^4 y \frac{1}{|\vec{y}|^4} (2\pi)^4 \sum_{\vec{n} \neq 0} \delta^4(\vec{x}-\vec{y}+2\pi\vec{n}) = -12\pi^2 \sqrt{g} \sum_{\vec{n} \in \mathcal{Z}^4 \neq 0} \frac{1}{|\vec{x}+2\pi\vec{n}|^5}
\end{aligned} \tag{C.5}$$

where the regularization consists of removing the $\vec{n} = 0$ term from the equality,

$$\sum_{\vec{p} \in \mathcal{Z}^4} e^{i\vec{p}\cdot\vec{x}} = (2\pi)^4 \sum_{\vec{n} \in \mathcal{Z}^4} \delta^4(\vec{x}+2\pi\vec{n}) \tag{C.6}$$

and the sum on \vec{n} is on the original lattice $\vec{n} = n^i \in \mathcal{Z}^4$. The regularized vacuum energy is

$$\langle H \rangle = -\frac{3}{16\pi^3} \sqrt{g} \sum_{\vec{n} \in \mathcal{Z}^4 \neq 0} \frac{1}{(g_{ij} n^i n^j)^{\frac{5}{2}}} = -6\pi^2 \sqrt{g} \sum_{\vec{n} \neq 0} \frac{1}{|2\pi\vec{n}|^5}. \tag{C.7}$$

For a d -dimensional lattice sum, the general formula used in [3] is

$$\begin{aligned}
|\vec{p}| &= 2\pi^{-\frac{d}{2}} \frac{\Gamma(\frac{d+1}{2})}{\Gamma(-\frac{1}{2})} \int d^d y \sqrt{G_d} e^{-i\vec{p}\cdot\vec{y}} \frac{1}{|\vec{y}|^{d+1}}, \\
\langle H \rangle &= \frac{1}{2} \sum_{\vec{p} \in \mathcal{Z}^d} |\vec{p}| e^{i\vec{p}\cdot\vec{x}}|_{\vec{x}=0} = \frac{1}{2} \sum_{\vec{p} \in \mathcal{Z}^d} \sqrt{g^{\alpha\beta} p_\alpha p_\beta} \\
&= 2^d \pi^{\frac{d}{2}} \frac{\Gamma(\frac{d+1}{2})}{\Gamma(-\frac{1}{2})} \sqrt{G_d} \sum_{\vec{n} \in \mathcal{Z}^d \neq 0} \frac{1}{|2\pi\vec{n}|^{d+1}}.
\end{aligned} \tag{C.8}$$

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